AN INVERSE PROBLEM FOR SCHRÖDINGER EQUATIONS WITH DISCONTINUOUS MAIN COEFFICIENT.

LUCIE BAUDOUIN AND ALBERTO MERCADO

ABSTRACT. This paper concerns the inverse problem of retrieving a stationary potential for the Schrödinger evolution equation in a bounded domain of $\mathbb{R}^N$ with Dirichlet data and discontinuous principal coefficient $a(x)$ from a single time-dependent Neumann boundary measurement. We consider that the discontinuity of a is located on a simple closed hyper-surface called the *interface*, and a is constant in each one of the interior and exterior domains with respect to this interface. We prove uniqueness and lipschitz stability for this inverse problem under certain convexity hypothesis on the geometry of the interior domain and on the sign of the jump of a at the interface. The proof is based on a global Carleman inequality for the Schrödinger equation with discontinuous coefficients, result also interesting by itself.

1. INTRODUCTION

The method of Carleman estimates was introduced in the field of inverse problems by BUKHGEIM and KLIBANOV in reference [8] (see also [7] and [17]). The first known results concern uniqueness of inverse problems. Then, one of the first stability result for a multidimensional inverse problem, dealing with an hyperbolic equation, can be read in [23] and is based on a modification of an idea of [8].

It is possible to obtain local Lipschitz stability around the single known solution, provided that this solution is regular enough and contains enough information [16] (see also [17] and [25]). Actually, many of the results using the same strategy we can refer to concern the wave equation. A complete list is too long to be given here but to cite some of them, related to the same kind of inverse problems of determining a potential, see [22] and [25] for a Dirichlet boundary data and a Neumann measurement and [14] for a Neumann boundary data and a Dirichlet measurement. These references are all based upon the use of local or global Carleman estimates.

Recently, global Carleman estimates and applications to one-measurement inverse problems were obtained in the case of variable but still regular coefficients, see [13] for the isotropic case, and [18] and [3] for the anisotropic case. It is interesting to note that these authors require a bound on the gradient of the coefficients, so that the idea of approximating discontinuous coefficients by smooth ones is not useful. Nevertheless, uniqueness and Lipschitz stability are obtained in [1] for the inverse problem of retrieving a stationary potential for the wave equation with

1991 *Mathematics Subject Classification.* 35R30, 35Q40.

Key words and phrases. Inverse problem, Carleman inequality, degenerate weight, pseudoconvexity, Schrödinger equation.

Dirichlet data and discontinuous principal coefficient from a single time-dependent Neumann boundary measurement.

One can also note that a global Carleman estimate has been obtained [11] for the heat equation with discontinuous coefficients. That work was initially motivated by the study of the exact null controllability of the semilinear heat equation, but the estimate has been recently used to prove local Lipschitz stability for a one measurement inverse problem. In this field, one should also read [4], [5] and [6].

Up to our knowledge, the result of determination of a time independent potential in Schrödinger evolution equation with discontinuous principal coefficient from a single time dependent measurement on the boundary is new. Concerning the simpler case of a "classical" Schrödinger equation (with $a = 1$), one can have a look at [2], where the Carleman estimate and the proof of the stability of the same inverse problem are maybe easier to read and the philosophy is the same. For the same equation, one can find in [21] a method with weight functions satisfying a relaxed pseudoconvexity condition, which allows to prove Carleman inequalities with less restrictive boundary observations than in [2]. The authors of [19] deal with Carleman estimates for the Schödinger equation with variable (but regular) principal coefficient and applications to controllability. Let us notice that in the different context of Cauchy problem, V. ISAKOV in [15] uses local Carleman estimates for the Schrödinger equation to prove uniqueness of the solution. Finaly, for the Schrödinger operator $i\partial_t + \operatorname{div}(c\nabla)$ in an unbounded strip in $\mathbb{R}^2$, reference [9] gives a stability result for the diffusion coefficient c in H^1 with only one observation in an unbounded domain. One will see in the proof of our main tool (an appropriate Carleman estimate) that it is based on the same strong pseudoconvexity condition (H_4) for the weight ψ.

1.1. **Statement of the problem and main results.** Let $T > 0$ and let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with C^2-boundary $\partial\Omega$. Throughout this paper, we use the following notations :

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_N}\right), \ \Delta v = \sum_{i=1}^{N} \frac{\partial^2 v}{\partial x_i^2} \ ,$$

$$v' = \frac{\partial v}{\partial t} \ \text{ and } \ v'' = \frac{\partial^2 v}{\partial t^2} \ ,$$

$\nu \in \mathbb{R}^N$ denotes the unit outward normal vector to $\partial\Omega$,

$\dfrac{\partial v}{\partial \nu} = \nabla v . \nu$ is the normal derivative.

We will work with the following Schrödinger equation :

$$\begin{cases} iy'(x,t) + \operatorname{div}(a(x)\nabla y(x,t)) + p(x)y(x,t) = 0, & x \in \Omega, \ t \in (0,T) \\ y(x,t) = h(x,t), & x \in \partial\Omega, \ t \in (0,T) \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{1}$$

We consider in this paper the inverse problem of the determination of the coefficient p of the lower order term in Schrödinger equation (1) from a single time dependent observation of Neumann data $\frac{\partial y}{\partial \nu}$ on the boundary.

The major novelty of this paper is that we deal with a Schrödinger equation in a bounded domain of $\mathbb{R}^N$ with discontinuous principal coefficient. Indeed, let Ω and Ω_1 be two open subsets of $\mathbb{R}^N$ with smooth boundaries Γ and Γ_1. We choose Ω_1 simply connected and such that $\overline{\Omega}_1 \subset \Omega$ and we set $\Omega_2 = \Omega \backslash \overline{\Omega}_1$. Thus, we have $\partial\Omega_2 = \Gamma \cup \Gamma_1$ and we also set:

$$a(x) = \begin{cases} a_1 & x \in \Omega_1 \\ a_2 & x \in \Omega_2 \end{cases}$$

with $a_j > 0$ for $j = 1, 2$.

Considering equation (1), we know that for each $p \in L^\infty(\Omega)$, $y_0 \in L^2(\Omega)$ and $h \in L^2(\Gamma \times (0,T))$, there exists a unique weak solution y such that

$$y \in C([0,T]; H^{-1}(\Omega)) \cap H^{-1}(0,T; L^2(\Omega)).$$

The proof is based on a transposition method, as one can read in [20]. Let us also notice that the regularity of y also implies $\dfrac{\partial y}{\partial \nu} \in H^{-2}(0,T; H^{-\frac{3}{2}}(\Gamma))$.

We will prove the well-posedness of the inverse problem consisting in retrieving the potential p involved in equation (1), knowing the flux (the normal derivative) of the solution $y(p)$ of (1) on the boundary. It means that we will prove uniqueness and stability of the nonlinear inverse problem characterized by the nonlinear application

$$p|_\Omega \longmapsto a_2 \frac{\partial y}{\partial \nu}\bigg|_{\Gamma \times (0,T)}. \tag{2}$$

We will more precisely answer the following questions.

Uniqueness :

Does the equality $\dfrac{\partial y(p)}{\partial \nu} = \dfrac{\partial y(q)}{\partial \nu}$ on $\Gamma \times (0,T)$ imply $p = q$ on Ω ?

Stability :

Is it possible to estimate $q - p|_\Omega$ by $\dfrac{\partial y(q)}{\partial \nu} - \dfrac{\partial y(p)}{\partial \nu}\bigg|_{\Gamma \times (0,T)}$ in suitable norms ?

Indeed, we will only give a local answer about the determination of p, working first on a linearized version of the problem, as shown is Section 3. Assuming that $p \in L^\infty$ is a given function, we are concerned with the stability around p. That is to say p and $u(p)$ are known while q is unknown. We can also add that *uniqueness* is a direct consequence of *stability* but historically, uniqueness results were obtain first (see [8]) and stability was proved using for instance compactness-uniqueness arguments as in [25].

In this work we introduce a Carleman weight whose spatial part is similar to the one of the weight function constructed in [1] for the two-dimensional case.

We prove a new Carleman estimate for the Shrödinger equation (see Theorem 9) under the hypothesis of **strong convexity** -also called uniform convexity- for the interface (roughly speaking, it means that their curvatures are uniformly bounded from below by a positive constant; see Definition 6), and some sign for the jump of the main coefficient. The following result, based on this Carleman estimate, states the stability of the inverse problem.

Theorem 1. *Assume that* Ω_1 *is strongly convex and* $a_1 > a_2 > 0$. *Let* $\mathcal{U}$ *be a bounded subset of* $L^\infty(\Omega)$, $p \in L^\infty(\Omega)$ *and* $r > 0$. *If* $y_0 \in H^1(\Omega)$ *is real valued (or pure imaginary) and if*

$$|y_0(x)| \geq r > 0, \ a.e. \text{ in } \Omega,$$

$$y(p) \in H^1(0, T; L^\infty(\Omega)),$$

then there exists $C = C(\Omega, T, \|p\|_{L^\infty(\Omega)}, \mathcal{U}) > 0$ *such that*

$$\|p - q\|_{L^2(\Omega)} \leq C \left\| a_2 \frac{\partial y(p)}{\partial \nu} - a_2 \frac{\partial y(q)}{\partial \nu} \right\|_{H^1(0,T;L^2(\Gamma))}$$

$q \in \mathcal{U}$, *where* $y(p)$ *is the solution of equation* (1) *with potential* p.

The main idea is that the *nonlinear inverse problem* is reduced to some *perturbed inverse problem* which will be solved with the help of a Carleman estimate. In order to obtain such an estimate, we first rewrite (1) as a system of two Schrödinger equations with constant coefficients and solutions y_1 and y_2, coupled with transmission conditions. We then construct a Carleman inequality on each domain with nonzero boundary values on the interface. Next, we gather all the terms to construct a global Carleman inequality for the transmission problem. The main point is to look carefully at the interaction of y_1 and y_2 on the common boundary Γ_1.

Notice that we state hypothesis for a function which guarantee that it would be a suitable weight function for a Carleman estimate with the only requirement that the discontinuities of a are located on Γ_1. We shall only construct an explicit weight function for the case of a discontinuous coefficient which is constant on each subdomain (i.e. a_1 and a_2 constants). However, we could also construct a weight function for variable coefficients $a_1(x)$ and $a_2(x)$ such that their traces at the interface are constant, under additional assumptions of boundedness of ∇a_j similar to those appearing in [13] (in order that the corresponding weight function would satisfy hypothesis (H_3) and (H_4) in section 2).

This article is organized as follows. Section 2 is devoted to the proof of an appropriate global Carleman inequality and Section 3 concerns the proof of the Lipschitz stability of our the inverse problem.

2. A GLOBAL CARLEMAN ESTIMATE

In this step, we will show a global Carleman estimate concerning a function $v = v(x, t)$ equals to zero on $\partial\Omega \times (-T, T)$ and solution of a Schrödinger equation with a bounded potential $q = q(x)$. We set the following notations :

$$Q = \Omega \times (-T, T) \qquad \Omega_0 = \Omega_1 \cup \Omega_2$$
$$\Gamma = \partial\Omega \qquad \Gamma_1 = \partial\Omega_1 \cap \partial\Omega_2$$
$$\Sigma = \Gamma \times (-T, T) \qquad \Sigma_1 = \Gamma_1 \times (-T, T)$$

and if u is a function defined in Ω, for u_j we will mean its restriction to the set Ω_j, for each $j = 1, 2$.

The main hypothesis for the Carleman estimate is the existence of a weight function $\psi = \psi(x)$ defined on $\mathbb{R}^N$ such that, on the one hand it is pseudo-convex with respect to the Schrödinger operator in each one of the two sub-domains Ω_1 and Ω_2, and on the other hand it has a convenient behavior at the interface Γ_1.

Indeed, we will first suppose that $\psi \in C^4(\overline{\Omega})$ verifies the natural transmission conditions:

$$\begin{cases} \psi_1 = \psi_2 & \text{on } \Gamma_1 \\ a_1 \dfrac{\partial \psi_1}{\partial \nu_1} + a_2 \dfrac{\partial \psi_2}{\partial \nu_2} = 0 & \text{on } \Gamma_1. \end{cases} \tag{Tr}$$

We will also suppose the following behavior at the interface

$$\psi(x) = cte \qquad \text{for all } x \in \Gamma_1, \tag{H_1}$$

$$\frac{\partial \psi_1}{\partial \nu_1} + \frac{\partial \psi_2}{\partial \nu_2} < 0 \qquad \text{on } \Gamma_1. \tag{H_2}$$

In the interior Ω_0 we will need that

$$|\nabla \psi| \geq \delta > 0 \tag{H_3}$$

and that $\exists\, \epsilon > 0$ such that

$$2D_a^2\psi(\xi, \bar{\xi}) + 2a^2\lambda|\nabla\psi \cdot \xi|^2 - a\nabla a \cdot \nabla\psi|\xi|^2 \geq \epsilon|\xi|^2 \tag{H_4}$$

$\forall \xi \in \mathbb{C}^n$, where

$$D_a^2\psi = \left(a \frac{\partial}{\partial x_i} \left(a \frac{\partial \psi}{\partial x_j} \right) \right)_{1 \leq i,j \leq N}.$$

Finally, it will be useful to consider weight functions satisfying (H_3) and (H_4) except in a neighborhood of a point. In this case we will need two weight functions ψ^1 and ψ^2, each one satisfying (H_3) and (H_4) in $\Omega_1 \cup \Omega_2 \setminus B_\varepsilon(x_1)$ and $\Omega_1 \cup \Omega_2 \setminus B_\varepsilon(x_2)$ respectively, (with ε small enough) such that

$$\psi^j - \psi^k \geq \delta > 0 \qquad \text{in } B_\varepsilon(x_k) \tag{H_5}$$

for each $j, k \in \{1, 2\}$ with $j \neq k$.

Summarizing, we set the following

Definition 2. (1) *Let $U \subset \Omega$ be an open set such that $\Gamma_1 \subset U$ and let $\psi \in C^4(U \setminus \Gamma_1)$. We say that ψ is a transmission weight function for equation* (1) *in U if it satisfies the conditions* (Tr)*, (H_1) and (H_2) on the interface Γ_1, and hypothesis (H_3) and (H_4) in U.*

(2) *Let ψ^1 and ψ^2 be two functions in $C^4(\Omega_1 \cup \Omega_2)$. We say that (ψ^1, ψ^2) is an ε-pair of transmission weight functions for (1) if there exist $x_1, x_2 \in \Omega_0$ and $\varepsilon > 0$ such that for each $k = 1, 2$ the function ψ^k is a transmission weight function for (1) in $\Omega_0 \setminus \overline{B_\varepsilon(x_k)}$ and the hypothesis (H_5) is fulfilled.*

Given ψ, for $s > 0$, $\lambda > 0$ we define on $Q = \Omega \times (-T, T)$ the following functions:

$$\theta(x,t) = \frac{e^{\lambda\psi(x)}}{(T-t)(T+t)} \quad \text{and} \quad \varphi(x,t) = \frac{\alpha - e^{\lambda\psi(x)}}{(T-t)(T+t)}$$

where $\alpha > \|e^{\lambda\psi}\|_{L^\infty(\Omega)}$.

We also define the space

$$Z = \left\{ v \in L^2(-T,T; H_0^1(\Omega)) : Lv \in L^2(Q), \frac{\partial v}{\partial \nu} \in L^2(\Sigma) \text{ and } v \text{ satisfies (Tr)} \right\},$$

introduce the following norm in Z

$$\|w\|_{s,\lambda,\psi} = s^3\lambda^4 \int_{-T}^{T}\int_\Omega \theta^3|w|^2 dxdt + s\lambda \int_{-T}^{T}\int_\Omega \theta|\nabla w|^2 dxdt \tag{3}$$

and for $\|\cdot\|_{s,\lambda,\psi,U}$ we will mean the above terms defined in the set $U \subset \Omega$.

We finally set

$$Lv = iv' + \text{ div}(a(x)\nabla u) + qv,$$
$$v = e^{s\varphi} w$$

and

$$Pw = e^{-s\varphi} L(e^{s\varphi} w).$$

Hence we have

$$\begin{aligned} Pw &= iw' + is\varphi' w + \text{ div}(a\nabla w) + 2sa\nabla\varphi.\nabla w \\ &\quad + sw \text{ div}(a\nabla\varphi) + s^2 a|\nabla\varphi|^2 w + qw \\ &= P_1 w + P_2 w + qw \end{aligned}$$

where we denoted

$$P_1 w = iw' + \text{ div}(a\nabla w) + s^2 a|\nabla\varphi|^2 w,$$
$$P_2 w = is\varphi' w + 2sa\nabla\varphi.\nabla w + s \text{ div}(a\nabla\varphi) w.$$

Our main result is the following

Theorem 3. *Suppose there exists for some $\varepsilon > 0$ an ε-pair of transmission weight functions (ψ^1, ψ^2) belonging to $C^3(\Omega_1 \cup \Omega_2)$. Let θ^k, φ^k and w^k be the corresponding functions defined for ψ^k as we did before. We also define*

$$\Sigma_+^{\psi^k} = \left\{ (x,t) \in \Gamma \times (-T,T) \; : \nabla\psi^k(x,t) \cdot \nu(x) > 0 \right\}$$

Then there exists $C > 0$, $s_0 > 0$ and $\lambda_0 > 0$ such that

$$\sum_{k=1}^{2}\left(\left\|P_1^{\psi^k}(w^k)\right\|^2_{L^2(Q)} + \left\|P_2^{\psi^k}(w^k)\right\|^2_{L^2(Q)} + \left\|w^k\right\|^2_{\lambda,s,\psi^k}\right)$$

$$\leq C\sum_{k=1}^{2}\left(\left\|P^{\psi^k}(w^k)\right\|^2_{L^2(Q)} + s\lambda\iint_{\Sigma_+^{\psi^k}}\theta^k\left|a\frac{\partial w^k}{\partial\nu}\right|^2\right) \tag{4}$$

for all $v \in Z$, $\lambda \geq \lambda_0$ and $s \geq s_0$.

2.1. **Formal computations.** We have

$$\begin{aligned}\iint_Q |Pw - qw|^2\,dxdt &= \iint_Q |P_1 w|^2\,dxdt + \iint_Q |P_2 w|^2\,dxdt \\ &\quad +2Re\iint_Q P_1 w\overline{P_2 w}\,dxdt,\end{aligned}$$

where $\overline{z}$ is the conjugate of z and $Re(z)$ its real part.
As $v \in L^2(-T,T;H_0^1(\Omega))$ and $v' \in L^2(-T,T;H^{-1}(\Omega))$ (because $Lv \in L^2(Q)$), we have $v \in C([-T,T];L^2(\Omega))$ and $w \in C([-T,T];L^2(\Omega))$ with $w(x,\pm T) = 0$.

We look for lower bounds for

$$Re\iint_Q P_1 w\overline{P_2 w}\,dxdt = \langle P_1 w, P_2 w\rangle_{L^2}$$

We set $\langle P_1 w, P_2 w\rangle_{L^2} = \sum_{i,j=1}^{3} I_{i,j}$, where $I_{i,j}$ is the integral of the product of the ith-term of $P_1 w$ and the jth-term of $P_2 w$. The properties of w and some integrations by parts allow to write the following equalities.

To begin with, we have

$$I_{11} = Re\iint_Q iw'(-is\varphi'\overline{w})\,dxdt = -\frac{s}{2}\iint_Q \varphi''|w|^2\,dxdt.$$

Applying the identity $Im(z) = -Im(\overline{z})$ for $z = 2s\lambda\displaystyle\int_{-T}^{T}\int_\Omega \theta a\nabla\psi\cdot\nabla\overline{w}w'\,dxdt$ we obtain:

$$\begin{aligned}I_{12} &= Re\iint_Q iw'(2sa\nabla\varphi\cdot\nabla\overline{w})\,dxdt \\ &= s\lambda\,Im\iint_Q \theta(\,\mathrm{div}(a\nabla\psi) + \lambda a|\nabla\psi|^2)w\overline{w'}\,dxdt \\ &\quad - s\lambda\,Im\iint_Q a\theta' w\nabla\psi\cdot\nabla\overline{w}\,dxdt \\ &\quad +s\lambda\iint_\Sigma \overline{w}a\theta w'\frac{\partial\psi}{\partial\nu}\,d\sigma dt\end{aligned}$$

We also have

$$\begin{aligned} I_{13} &= Re \iint_Q iw' s \operatorname{div}(a\nabla\varphi)\overline{w}\,dxdt \\ &= -s\lambda\, Im \iint_Q \theta(\operatorname{div}(a\nabla\psi) + \lambda a|\nabla\psi|^2) w\overline{w}' dxdt, \end{aligned}$$

$$\begin{aligned} I_{21} &= Re \iint_Q \operatorname{div}(a\nabla w)(-is\varphi'\overline{w})\,dxdt \\ &= s\lambda\, Im \iint_Q a\theta'\overline{w}\nabla\psi\cdot\nabla w\,dxdt + s\, Im \iint_\Sigma \varphi'\overline{w}a\frac{\partial w}{\partial\nu}\,d\sigma dt, \end{aligned}$$

and

$$\begin{aligned} I_{22} &= Re \iint_Q \operatorname{div}(a\nabla w)(2sa\nabla\varphi\cdot\nabla\overline{w})\,dxdt \\ &= -s\lambda \iint_Q \theta a|\nabla w|^2(\operatorname{div}(a\nabla\psi) + \lambda a|\nabla\psi|^2)\,dxdt \\ &\quad - s\lambda \iint_Q \theta a|\nabla w|^2\nabla a\cdot\nabla\psi\,dxdt + 2s\lambda^2 \iint_Q \theta a^2|\nabla\psi.\nabla w|^2\,dxdt \\ &\quad + 2s\lambda\, Re \iint_Q \theta D_a^2(\psi)(\nabla w, \nabla\overline{w})\,dxdt \\ &\quad - 2s\lambda \iint_\Sigma \theta a^2(\nabla\psi\cdot\nabla\overline{w})\frac{\partial w}{\partial\nu}\,d\sigma dt + s\lambda \iint_\Sigma \theta a^2|\nabla w|^2\frac{\partial\psi}{\partial\nu}\,d\sigma dt. \end{aligned}$$

Using integrations by parts we obtain

$$\begin{aligned} I_{23} &= Re \iint_Q \operatorname{div}(a\nabla w)s\operatorname{div}(a\nabla\varphi)\overline{w}\,dxdt \\ &= s\lambda \iint_Q |\nabla w|^2\theta(a\operatorname{div}(a\nabla\psi) + \lambda|a\nabla\psi|^2)\,dxdt \\ &\quad - \frac{s\lambda}{2} \iint_Q |w|^2 \operatorname{div}(\theta a\nabla\operatorname{div}(a\nabla\psi))\,dxdt \\ &\quad + \frac{s\lambda}{2} \iint_\Sigma |w|^2\theta a\nabla\operatorname{div}(a\nabla\psi)\cdot\nu\,d\sigma dt \\ &\quad - \frac{s\lambda^2}{2} \iint_Q |w|^2(\operatorname{div}(a\theta\nabla(a|\nabla\psi|^2)) + \operatorname{div}(a\theta\operatorname{div}(a\nabla\psi)\nabla\psi))\,dxdt \\ &\quad + \frac{s\lambda^2}{2} \iint_\Sigma |w|^2\theta a(\operatorname{div}(a\nabla\psi)\nabla\psi + \nabla(a|\nabla\psi|^2))\cdot\nu\,d\sigma dt \\ &\quad - \frac{s\lambda^3}{2} \iint_Q |w|^2\operatorname{div}(a^2\theta|\nabla\psi|^2\nabla\psi)\,dxdt + \frac{s\lambda^3}{2} \iint_\Sigma |w|^2a^2\theta|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}\,d\sigma dt \\ &\quad - \lambda s Re \iint_\Sigma \overline{w}\theta a(\operatorname{div}(a\nabla\psi) + \lambda|\nabla\psi|^2 a)\frac{\partial w}{\partial\nu}\,d\sigma dt. \end{aligned}$$

and we obviously have

$$I_{31} \quad = \quad Re \iint_Q s^2 a|\nabla\varphi|^2 w(-is\varphi'\overline{w})\,dxdt = 0,$$

$$\begin{aligned}
I_{32} \quad &= \quad Re \iint_Q s^2 a|\nabla\varphi|^2 w(2sa\nabla\varphi \cdot \nabla\overline{w})\,dxdt \\
&= \quad s^3\lambda^3 \iint_Q |w|^2\theta^3 a\left(|\nabla\psi|^2 \operatorname{div}(a\nabla\psi) + 2aD^2(\psi)(\nabla\psi,\nabla\psi)\right)\,dxdt \\
&\quad + s^3\lambda^3 \iint_Q |w|^2\theta^3 a|\nabla\psi|^2\nabla a\cdot\nabla\psi\,dxdt \\
&\quad + 3s^3\lambda^4 \iint_Q |w|^2\theta^3 a^2|\nabla\psi|^4\,dxdt - \lambda^3 s^3 \iint_\Sigma |w|^2 a^2|\nabla\psi|^2\theta^3\frac{\partial\psi}{\partial\nu}\,d\sigma dt,
\end{aligned}$$

and

$$\begin{aligned}
I_{33} \quad &= \quad Re \iint_Q s^2 a|\nabla\varphi|^2 w(s\operatorname{div}(a\nabla\varphi)\overline{w})\,dxdt \\
&= \quad - s^3\lambda^3 \iint_Q |w|^2\theta^3 a|\nabla\psi|^2 \operatorname{div}(a\nabla\psi)\,dxdt \\
&\quad - s^3\lambda^4 \iint_Q |w|^2\theta^3 a^2|\nabla\psi|^4\,dxdt.
\end{aligned}$$

Then we have $Re \iint_Q P_1 w\overline{P_2 w}\,dxdt \;=\; F(w) + G(\nabla w) + J + X_1$ where we define

$$F(w) \quad = \quad 2s^3\lambda^4 \iint_Q |w|^2\theta^3 a^2|\nabla\psi|^4\,dxdt, \tag{5}$$

$$\begin{aligned}
G(\nabla w) \quad &= \quad 2s\lambda^2 \iint_Q \theta a^2|\nabla\psi\cdot\nabla w|^2\,dxdt + 2s\lambda Re \iint_Q \theta D_a^2\psi(\nabla w,\nabla\overline{w})\,dxdt \\
&\quad -s\lambda \iint_Q |\nabla w|^2\theta a\nabla a\cdot\nabla\psi\,dxdt, \qquad (6)
\end{aligned}$$

J as the sum of all boundary integrals

$$\begin{aligned} J \;=\; & s\lambda Im \iint_\Sigma a\theta w'\overline{w}\frac{\partial\psi}{\partial\nu}\,d\sigma dt \\ & sIm\iint_\Sigma \varphi'\overline{w}a\frac{\partial w}{\partial\nu}\,d\sigma dt \\ & -2s\lambda Re\iint_\Sigma a^2\theta\nabla\psi\cdot\nabla\overline{w}\frac{\partial w}{\partial\nu}\,d\sigma dt \\ & +s\lambda\iint_\Sigma a^2\theta|\nabla w|^2\frac{\partial\psi}{\partial\nu}\,d\sigma dt \\ & -s\lambda Re\iint_\Sigma \overline{w}\,\mathrm{div}(a\theta\nabla\psi)a\frac{\partial w}{\partial\nu}\,d\sigma dt \\ & -s^3\lambda^3\iint_\Sigma a^2\theta^3|w|^2|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}\,d\sigma dt \\ & +\frac{s\lambda^3}{2}\iint_\Sigma a^2\theta|w|^2|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}\,d\sigma dt \\ & +\frac{s\lambda^2}{2}\iint_\Sigma a^2\theta|w|^2\nabla(|\nabla\psi|^2)\cdot\nu\,d\sigma dt \\ & +\frac{s\lambda^2}{2}\iint_\Sigma a\theta|w|^2\,\mathrm{div}(a\nabla\psi)\frac{\partial\psi}{\partial\nu}\,d\sigma dt \\ & +\frac{s\lambda}{2}\iint_\Sigma a\theta|w|^2\nabla(\,\mathrm{div}(a\nabla\psi))\cdot\nu\,d\sigma dt \end{aligned}$$

and X_1 as the sum of all the remaining integrals in Ω.

Moreover, if $U\subset\Omega$ is an open set, we will write $F_U(w)$ to denote the sum of integrals from the definition of $F(w)$ taken in the set U, and the same for G, X_1...

Noticing that

- $2s\lambda\, Im\iint_Q \theta' aw\nabla\psi.\nabla\overline{w}\,dxdt$
  $\le s\lambda\iint_Q a^2(\theta')^{\frac{1}{2}}|\nabla\psi.\nabla w|^2\,dxdt + \; s\lambda\iint_Q(\theta')^{\frac{3}{2}}|w|^2\,dxdt,$
- $a\in W^{2,\infty}(\Omega)$ and $\psi\in C^4(\overline{\Omega})$
- $|\theta|\le C\theta^3,\ |\theta'|\le C\theta^2$ and$|\varphi''|\le C\theta^3$ on $(-T,T)\times\Omega,\ C=C(T)>0.$

it is then easy to prove, from simple calculations, that the "negligible" terms X_1 indeed satisfy

$$|X_1|\le Cs\lambda\iint_Q a^2\theta|\nabla\psi\cdot\nabla w|^2 + Cs\lambda^4\iint_Q\theta|w|^2 + Cs^3\lambda^3\iint_Q\theta^3|w|^2. \quad (7)$$

2.2. **Proof of the Carleman estimate.** In this part of the paper, we prove Theorem 3. We apply the above computations in each one of the domains Ω_1 and Ω_2 and

we sum up all the terms. Since the interface Γ_1 has null $\mathbb{R}^N$-measure, we get an estimate in all the set Ω, plus the boundary terms from $\partial\Omega$, and from the interface itself, where appear terms coming from both Ω_1 and Ω_2.
Given the hypothesis we have assumed, we prove in the following propositions that we can deal with all this terms. In the sequel, C denotes a generic constant, depending on T and Ω.

2.2.1. *The interior.* Recall the norm $\|\cdot\|_{s,\lambda,\psi}$ defined in (3) and $F(w)$, $G(\nabla w)$ defined in (5), (6).

Proposition 4. *Suppose that $U \subset \Omega$ is a open set and ψ satisfies (H_3) and (H_4) in $\Omega_1 \cup \Omega_2 \setminus U$. Then there exist $\gamma > 0$, $C \in \mathbb{R}$, s_0 and λ_0 such that for all $v \in Z$,*

$$F(w) + G(\nabla w) + X_1 \geq \gamma \|w\|_{s,\lambda,\psi} - C\|w\|_{s,\lambda,\psi,U}$$

$\forall s \geq s_0$ *and* $\forall \lambda \geq \lambda_0$.

Proof :
First, merely by the fact that $\psi \in C^4(\overline{\Omega})$, we have

$$|F_U(w)| + |G_U(\nabla w)| + |X_{1,U}| \leq C\|w\|_{s,\lambda,\psi,U} \tag{8}$$

for all $v \in Z$.
Now, from (7) and ψ satisfying (H_3) we get that for s and λ large enough,

$$|X_1| \leq s\lambda^2 \iint_Q \theta a^2 |\nabla\psi \cdot \nabla w|^2 + s^3\lambda^4 \iint_Q |w|^2 \theta^3 a^2 |\nabla\psi|^4$$

Hence, If ψ satisfies (H_3) and (H_4) in $\Omega_* = \Omega_1 \cup \Omega_2 \setminus U$ we get that $\forall\lambda \geq \lambda_0$, $s \geq s_0$ and $v \in Z$,

$$\begin{aligned}
F_{\Omega_*}(w) \quad + &\quad G_{\Omega_*}(\nabla w) + X_{1,\Omega_*} \\
\geq &\quad F_{\Omega_*}(w) + G_{\Omega_*}(\nabla w) - |X_{1,\Omega_*}| \\
\geq &\quad F_{\Omega_*}(w) + G_{\Omega_*}(\nabla w) - s\lambda^2 \int_{-T}^{T}\int_{\Omega_*} \theta a^2 |\nabla\psi \cdot \nabla w|^2 \\
&\quad - s^3\lambda^4 \int_{-T}^{T}\int_{\Omega^*} |w|^2\theta^3 a^2 |\nabla\psi|^4 \\
\geq &\quad s^3\lambda^4 \int_{-T}^{T}\int_{\Omega_*} |w|^2\theta^3 a^2 |\nabla\psi|^4 + s\lambda^2 \int_{-T}^{T}\int_{\Omega_*} \theta a^2 |\nabla\psi \cdot \nabla w|^2 \\
&\quad + 2s\lambda \mathrm{Re} \int_{-T}^{T}\int_{\Omega_*} \theta D_a^2\psi(\nabla w, \nabla\bar{w}) - s\lambda \int_{-T}^{T}\int_{\Omega_*} |\nabla w|^2 \theta a \nabla a \cdot \nabla\psi \\
\geq &\quad s^3\lambda^4 \int_{-T}^{T}\int_{\Omega_*} |w|^2\theta^3 a^2 |\nabla\psi|^4 + \epsilon s\lambda \int_{-T}^{T}\int_{\Omega_*} \theta |\nabla w|^2 \\
\geq &\quad \gamma \|w\|_{s,\lambda,\psi,\Omega_*}.
\end{aligned} \tag{9}$$

From (8) and (9) we get the desired result. ■

2.2.2. *The boundary.* By definition we have $w = 0$ on the exterior boundary Σ for each $v \in Z$. Therefore, $\nabla w|_\Sigma = \dfrac{\partial w}{\partial \nu}\nu$ and if we choose the legitimate notation $J = J_\Sigma + J_{\Sigma_1}$, we get here

$$\begin{aligned}
J_\Sigma &= -2s\lambda Re \iint_\Sigma a^2\theta \nabla\psi \cdot \nabla\overline{w}\frac{\partial w}{\partial \nu}\, d\sigma dt + s\lambda \iint_\Sigma a^2\theta|\nabla w|^2 \frac{\partial \psi}{\partial \nu}\, d\sigma dt \\
&= -s\lambda \iint_\Sigma \theta \left| a\frac{\partial w}{\partial \nu}\right|^2 \frac{\partial \psi}{\partial \nu}\, d\sigma dt \\
&\geq -s\lambda \iint_{\Sigma_+} \theta \left| a\frac{\partial w}{\partial \nu}\right|^2 \frac{\partial \psi}{\partial \nu}\, d\sigma dt \\
&\geq -s\lambda \left\| \frac{\partial \psi}{\partial \nu}\right\|_{L^\infty(\Sigma)} \iint_{\Sigma_+} \theta \left| a\frac{\partial w}{\partial \nu}\right|^2 d\sigma dt \\
&\geq -s\lambda C \iint_{\Sigma_+} \theta \left| a\frac{\partial w}{\partial \nu}\right|^2 d\sigma dt \qquad (10)
\end{aligned}$$

where we have denoted $\Sigma_+ = \{(x,t) \in \Gamma \times (-T,T) \,:\, \nabla\psi(x,t)\cdot\nu(x) > 0\}$.

2.2.3. *The interface.* We compute the sum of the integrals on the interface Σ_1, We write $J_{\Sigma_1} = \sum_{k=1}^{10} J_k$, enumerating the terms in the same order of the list in (7). For each $k = 1, \ldots, 10$ we denote as $[J_k]$ the sum of the k-term coming from the integrations by parts in Ω_1 with the corresponding one from Ω_2.

Proposition 5. *If ψ satisfies hypothesis (H_1), (H_2) and* (Tr) *then there exist λ_0 and s_0 such that*

$$J_{\Sigma_1} = \sum_{k=1}^{10}(J_k(w_1) + J_k(w_2)) \geq 0 \qquad (11)$$

for all $v \in Z$, $\forall\lambda \geq \lambda_0$, $s \geq s_0$.

Proof :

It is not difficult to check that $[J_k] = 0$ for $k = 1, 2$ since ψ and w satisfy the transmission conditions (Tr). Moreover, ψ is constant on the interface and then we obtain $\nabla\psi \cdot \nabla\overline{w} = \dfrac{\partial \psi}{\partial \nu}\dfrac{\partial \overline{w}}{\partial \nu}$ on Γ_1. Therefore, thanks to (H_2) we get

$$\begin{aligned}
[J_3] &= -2s\lambda \iint_{\Sigma_1} \theta \left| a_1\frac{\partial w_1}{\partial \nu_1}\right|^2 \left(\frac{\partial \psi_1}{\partial \nu_1} + \frac{\partial \psi_2}{\partial \nu_2}\right) d\sigma dt \\
&\geq s\lambda\delta \iint_{\Sigma_1} \theta \left| a_1\frac{\partial w_1}{\partial \nu_1}\right|^2 d\sigma dt.
\end{aligned}$$

By mean of the orthogonal decomposition $\nabla w = \dfrac{\partial w}{\partial \nu}\nu + \nabla_\tau w$, where $\nabla_\tau w$ is the projection of ∇w on the tangent hyper-plane of $\partial\Omega_1$, and from hypothesis (H_2)

and (H_3) and the fact that $\nabla_\tau w_1 = \nabla_\tau w_2$ we obtain

$$\begin{aligned}
[J_4] &= s\lambda \iint_{\Sigma_1} \theta \left| a_1 \frac{\partial w_1}{\partial \nu_1} \right|^2 \left(\frac{\partial \psi_1}{\partial \nu_1} + \frac{\partial \psi_2}{\partial \nu_2} \right) d\sigma dt \\
&\quad + s\lambda \iint_{\Sigma_1} \theta \left| \nabla_\tau w \right|^2 \left(a_1^2 \frac{\partial \psi_1}{\partial \nu_1} + a_2^2 \frac{\partial \psi_2}{\partial \nu_2} \right) d\sigma dt \\
&= s\lambda \iint_{\Sigma_1} \theta \left| a_1 \frac{\partial w_1}{\partial \nu_1} \right|^2 \left(\frac{\partial \psi_1}{\partial \nu_1} + \frac{\partial \psi_2}{\partial \nu_2} \right) d\sigma dt \\
&\quad + s\lambda \iint_{\Sigma_1} \theta \left| \nabla_\tau w \right|^2 a_1 a_2 \left(-\frac{\partial \psi_1}{\partial \nu_1} - \frac{\partial \psi_2}{\partial \nu_2} \right) d\sigma dt \\
&\geq s\lambda \iint_{\Sigma_1} \theta \left| a_1 \frac{\partial w_1}{\partial \nu_1} \right|^2 \left(\frac{\partial \psi_1}{\partial \nu_1} + \frac{\partial \psi_2}{\partial \nu_2} \right) d\sigma dt \\
&\geq 2s\lambda\delta \iint_{\Sigma_1} \theta \left| a_1 \frac{\partial w_1}{\partial \nu_1} \right|^2 d\sigma dt
\end{aligned}$$

$$\begin{aligned}
[J_6] &= -s^3\lambda^3 \iint_{\Sigma_1} \theta^3 |w_1|^2 \left| a_1 \frac{\partial \psi_1}{\partial \nu_1} \right|^2 \left(\frac{\partial \psi_1}{\partial \nu_1} + \frac{\partial \psi_2}{\partial \nu_2} \right) d\sigma dt \\
&\geq s^3\lambda^3\delta^2 \iint_{\Sigma_1} \theta^3 a_1^2 |w_1|^2 \, d\sigma dt
\end{aligned}$$

Since $a \in W^{2,\infty}(\Omega)$ and $\varphi \in C^4(\overline{\Omega})$, we also have

$$|[J_5]| \leq Cs^2\lambda^3 \iint_{\Sigma_1} |w_1|^2 \theta^3 \, d\sigma dt + C\lambda \iint_{\Sigma_1} \left| a \frac{\partial w_1}{\partial \nu_1} \right|^2 \theta \, d\sigma dt$$

and

$$\left| \sum_{k=7}^{10} [J_k] \right| \leq Cs^2\lambda^3 \iint_{\Sigma_1} |w_1|^2 \theta^3 \, d\sigma dt.$$

Thus, for s large enough, we get the desired result

$$\sum_{k=4}^{10} [J_k] \geq (s\delta - C) \left(s^2\lambda^3 \iint_{\Sigma_1} |w_1|^2 \theta^3 + \lambda \iint_{\Sigma_1} |a \frac{\partial w_1}{\partial \nu_1}|^2 \theta \right) \geq 0$$

■

2.2.4. *Carrying all together.* From (10) and Propositions 4 and 5 we obtain

$$\|w\|^2_{s,\lambda,\psi} - C\|w\|^2_{s,\lambda,\psi,U} - s\lambda C \iint_{\Sigma_+} \theta \left| a \frac{\partial w}{\partial \nu} \right|^2 \leq C Re \left\langle P_1(w), P_2(w) \right\rangle_{L^2} \quad (12)$$

Adding $\dfrac{C}{2}\left(|P_1(w)|^2_{L^2}+|P_2(w)|^2_{L^2}\right)$ to both sides of (12) we obtain

$$\frac{C}{2}\left(|P_1(w)|^2_{L^2}+|P_2(w)|^2_{L^2}\right)+\|w\|^2_{s,\lambda,\psi} - C\|w\|^2_{s,\lambda,\psi,U} - s\lambda C\iint_{\Sigma_+}\theta\left|a\frac{\partial w}{\partial\nu}\right|^2 \leq C\,|P(w)-qw|^2_{L^2},$$

what means that for all $s\geq s_2$ and $\lambda\geq\lambda_2$, since $C>0$ is a generic constant,

$$|P_1(w)|^2_{L^2}+|P_2(w)|^2_{L^2}+\|w\|^2_{s,\lambda,\psi} \leq C|P(w)-qw|^2_{L^2}+C\|w\|^2_{s,\lambda,\psi,U}+s\lambda C\iint_{\Sigma_+}\theta\left|a\frac{\partial w}{\partial\nu}\right|^2. \quad (13)$$

Now, if (ψ^1,ψ^2) is an ε-pair of transmission weight functions (see Definition 2), we have an estimate like (13) for each ψ^k with $U=B_\varepsilon(x_k)$ where $x_j\in\Omega$, $j=1,2$ and $\varepsilon>0$.

We sum up both estimates and we can show that the left hand side of each inequality can absorb the right hand side term $\|\cdot\|_{s,\lambda,\psi^k,B_\varepsilon(x_k)}$ from the other inequality provided that ε is small and λ is large enough. Indeed, by assumption we have that $\psi^2-\psi^1>\delta>0$ in $B_\varepsilon(x_1)$.
Then, by taking λ large enough we have

$$e^{\lambda(\psi^2-\psi^1)}>2C \qquad \text{in} \quad B_\varepsilon(x_1)$$

i.e.

$$C\theta^1<\frac{1}{2}\theta^2 \qquad \text{in} \quad B_\varepsilon(x_1)$$

and we conclude that $\|w^1\|_{\psi^1,B_\varepsilon(x_1)}$ of the right and side is absorbed by the term $\|w^2\|_{\psi^2}$ of the left hand side. It is clear that an analogous result is true by interchanging ψ^1 and ψ^2. Theorem 3 is proved. ∎

2.3. **Particular case.** In this part of the work, we construct explicit weight functions adapted to particular discontinuous coefficients.

We need the following definition.

Definition 6. *We say that the open, bounded and convex set $U\subset\mathbb{R}^N$ ($N\geq 2$) is* **strongly convex** *if ∂U is of class C^2 and all the principal curvatures are strictly positive functions on ∂U.*

Remark 1. *Let us note that $U\subset\mathbb{R}^N$ is strongly convex if and only if for all plane $\Pi\subset\mathbb{R}^N$ intersecting U, the curve $\Pi\cap\partial U$ has strictly positive curvature at each point. In particular, a strongly convex set is geometrically strictly convex.*

We assume that $\Omega_1\subset\Omega$ is a strongly convex domain with boundary Γ_1 of class C^3, and we set $\Omega_2=\Omega\setminus\overline{\Omega}_1$. Thus, we have $\partial\Omega_2=\Gamma\cup\Gamma_1$, where this is a disjoint union. We deal with the case where a is locally constant

$$a(x)=\begin{cases} a_1 & x\in\Omega_1\\ a_2 & x\in\Omega_2\end{cases} \quad (14)$$

with $a_j > 0$ for $j = 1, 2$.

In order to construct a convenient weight function, take $x_0 \in \Omega_1$ and for each $x \in \Omega \setminus \{x_0\}$ define $\ell(x_0, x) = \{x_0 + \lambda(x - x_0) \ : \ \lambda \geq 0\}$. Since Ω_1 is convex there is exactly one point $y(x)$ such that $y(x) \in \Gamma_1 \cap \ell(x_0, x)$. Thus, we can define the function $\rho : \Omega \setminus \{x_0\} \longrightarrow \mathbb{R}^+$ by:

$$\rho(x) = |x_0 - y(x)|. \tag{15}$$

Let $\varepsilon > 0$ be such that $\overline{B}_\varepsilon \subset \Omega_1$ (and small enough in a sense we will precise later) and let $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$. Then we consider a cut-off function $\eta \in C^\infty(\mathbb{R}^N)$ such that

$$0 \leq \eta \leq 1, \qquad \eta = 0 \text{ in } B_{\varepsilon_1}(x_0), \qquad \eta = 1 \text{ in } \Omega \setminus \overline{B}_{\varepsilon_2}(x_0).$$

For each $j \in \{1, 2\}$ we take k such that $\{j, k\} = \{1, 2\}$ and we define the following functions in the whole domain Ω

$$\psi_j(x) = \eta(x) \frac{a_k}{\rho(x)^2} |x - x_0|^2 + M_j \qquad x \in \Omega,$$

where M_1 and M_2 are positive numbers such that

$$a_1 - a_2 = M_1 - M_2. \tag{16}$$

Then, the weight function we will use in this work is

$$\psi(x) = \begin{cases} \psi_1(x) & x \in \Omega_1 \\ \psi_2(x) & x \in \Omega_2. \end{cases} \tag{17}$$

Throughout the paper, we will use the notations $\bar{a}(x) = a_2 \mathbf{1}_{\Omega_1}(x) + a_1 \mathbf{1}_{\Omega_2}(x)$ and $M = M_1$ on Ω_1 and $M = M_2$ on Ω_2, so that we can write

$$\psi(x) = \eta(x)\bar{a}(x) \frac{|x - x_0|^2}{\rho(x)^2} + M.$$

As we can see in the following result, the main property of the weight function is a consequence of the strong convexity of the interior domain Ω_1.

Lemma 7. *If $\Omega_1 \subset \mathbb{R}^N$ is strongly convex and if the function $\mu : \mathbb{R}^N \longrightarrow \mathbb{R}^+$ is defined by $\mu(x) = \dfrac{|x - x_0|}{\rho(x)}$ then $D^2\mu^2(x)$ is positive definite for all $x \in \mathbb{R}^N \backslash \{x_0\}$, uniformly in bounded subsets of $\mathbb{R}^N \backslash \{x_0\}$.*

Proof :

We shall deduce this Lemma from well-known properties of compact convex subsets of $\mathbb{R}^N$ (called *convex bodies*). However, for the sake of completeness of this paper, we include in the Appendix a self-contained proof of this result.

Assuming without lost of generality that $x_0 = 0$, it is not difficult to see that μ is the gauge function of the convex set $\overline{\Omega_1}$ (in other words, μ is a seminorm whose unit ball is Ω_1, see [24], p. 43, and section 2.3 of [10]).

The proof that μ is a convex function of class C^2 (hence $D^2\mu \geq 0$) can be read in [10] (Theorem 2.1). Moreover, it is proved that for each $x \in \mathbb{R}^N \backslash \{0\}$ the only null eigenvalue of $D^2\mu(x)$ corresponds to the direction x (which is the radial direction). The others eigenvalues, as functions of x, are bounded below by a positive constant, uniformly in $x \neq 0$ given in a bounded subset of $\mathbb{R}^N$.

Thus, there exists $\delta > 0$ such that for all $x \in \overline{\Omega}$ we have

$$D^2\mu(x)(v,v) \geq \delta |v|^2 \ \ \forall\, v \in x^{\perp} = \{y \in \mathbb{R}^n \ : \ y \cdot x = 0\}. \tag{18}$$

On the other hand, we have $\nabla\mu(x) = \dfrac{1}{\rho(x)}\dfrac{x}{|x|} + |x|\nabla\dfrac{1}{\rho}(x)$. Since ρ is constant in the radial direction, we get $x \cdot \nabla\dfrac{1}{\rho}(x) = 0$. Hence we deduce that

$$x \cdot \nabla\mu(x) = \frac{|x|}{\rho(x)} = \mu(x) \neq 0. \tag{19}$$

Take $x, v \in \mathbb{R}^N \backslash \{0\}$. Then $v = v_1 \frac{x}{|x|} + v_2 y$, where y is an unitary element of $x^{\perp}$. In view of the fact that $D^2\mu^2(v,v) = 2\mu D^2\mu(v,v) + 2|v \cdot \nabla\mu|^2$, from (18), (19) and $D^2\mu(x)(x,x) = 0$ we get

$$\begin{aligned} D^2\mu^2(x)(v,v) &\geq 2\mu(x)D^2\mu(x)(v_2y, v_2y) + 2\left|v_1\frac{x}{|x|}\cdot\nabla\mu(x)\right|^2 \\ &\geq 2\mu(x)\delta v_2^2 + 2\frac{v_1^2}{|x|^2}\mu^2(x) \\ &= 2\mu(x)\delta v_2^2 + \frac{2}{\rho(x)^2}v_1^2 \\ &\geq \delta_1(v_1^2 + v_2^2) \end{aligned}$$

and we conclude that $D^2\mu^2(x)$ is positive definite.

■

Assuming the additional hypothesis about the sign of the jump on the interface, we can prove that the functions we have defined work as a weight function:

Proposition 8. *Let Ω_1 be an open and bounded set in $\mathbb{R}^N$ with smooth boundary, and a_1, a_2 real numbers such that:*

(1) *Ω_1 is strongly convex.*
(2) $0 < a_2 < a_1$.

Then, for each pair of points $x_1, x_2 \in \Omega_1$, there exists $\varepsilon > 0$ such that the above construction gives up an ε-pair of transmission weight functions (ψ^1, ψ^2) in the sense of Definition 2.

Proof :

For $x_0 \in \Omega_1$ let ψ be the function constructed as above and defined by (17). If $x \in \Gamma_1$ we have $\rho(x) = |x - x_0|$ and $\psi_j(x) = a_k + M_j$. From (16) we get $\psi_1 = \psi_2 = c$ on Γ_1. Moreover, if $x \in \Gamma_1$ we have

$$a_1\nabla\psi_1(x) = a_1a_2\nabla\left(\frac{|x-x_0|^2}{\rho(x)^2}\right) = a_2\nabla\psi_2(x).$$

Hence (Tr) and (H_1) are satisfied (recall that $\nu_1 = -\nu_2$ on Γ_1).

On the other hand, since Γ_1 is a level set of ψ_1, then $\psi_1(x) < a_2 + M_1 < \psi_1(y)$ for any $x \in \Omega_1$ and $y \in \Omega_2$, and we have $\dfrac{\partial \psi_1}{\partial \nu_1} > 0$ on Σ_1 and

$$\frac{\partial \psi_1}{\partial \nu_1} + \frac{\partial \psi_2}{\partial \nu_2} = \frac{\partial \psi_1}{\partial \nu_1}\left(1 - \frac{a_1}{a_2}\right) < 0$$

what gives (H_2).

For $x \in \Omega_0 \setminus B_\varepsilon(x_0)$, denoting $c(x) = \frac{\bar{a}}{\rho^2(x)}$, we get $\nabla\psi = 2c(x)(x - x_0) + |x - x_0|^2 \nabla c(x)$. By construction $c(x)$ is constant in the direction of $x - x_0$, hence

$$(x - x_0) \cdot \nabla c(x) = 0$$

and then

$$\begin{aligned} |\nabla\psi|^2 &= 4c^2(x)|x - x_0|^2 + |x - x_0|^4 |\nabla c(x)|^2 \\ &\geq 4c^2(x)|x - x_0|^2. \end{aligned}$$

Thus we have

$$|\nabla\psi|^2 \geq 4\left(\frac{\bar{a}}{\text{diam}(\Omega)^2}\right)^2 \varepsilon^2.$$

in $\Omega_0 \setminus B_\varepsilon(x_0)$ and ψ satisfies (H_3) in that set.

Property (H_4) is deduced from Lemma 7.

One can notice that x_0 can be arbitrarily chosen in Ω_1 since it is convex. Therefore, we can take two different points x_1, x_2 in Ω_1 and we can construct the respective weight functions ψ^1 and ψ^2. For each $k = 1, 2$, ψ^k is a transmission weight function in $\Omega_0 \setminus \overline{B_\varepsilon(x_k)}$ and it remains to be shown that (H_5) is fulfilled in order to finish the proof of Proposition 8.

Let be $d = \frac{1}{2}|x_1 - x_2|$ such that $\varepsilon < d$. On the one hand, for all $x \in B_\varepsilon(x_1)$ we have:

$$\psi^1(x) \leq \frac{a}{\rho_1^2}\varepsilon^2 + M \leq \frac{a}{\alpha_1^2}\varepsilon^2 + M,$$

where $\alpha_1 = d(x_1, \Gamma_1) > 0$. On the other hand, if we denote $D_2 = \max_{y \in \Gamma_1} d(y, x_2)$, we get, for all $x \in B_\varepsilon(x_1)$,

$$\psi^2(x) \geq \frac{a}{\rho_2^2}d^2 + M \geq \frac{a}{D_2^2}d^2 + M.$$

Consequently, we have

$$\psi^2 - \psi^1 \geq a\left(\frac{d^2}{D_2^2} - \frac{\varepsilon^2}{\alpha_1^2}\right) \qquad \forall x \in B_\varepsilon(x_1). \tag{20}$$

It is clear that an analogous result is true by interchanging x_1 and x_2 (now with α_2 and D_1). Thus, taking $\varepsilon < \min\left(\frac{d\alpha_1}{D_2}, \frac{d\alpha_2}{D_1}\right)$ we get (H_5) and Proposition 8 is proved. ■

¿From Proposition 8 and Theorem 3, we obtain the following result:

Theorem 9. *Let the coefficient a be constant in the open set Ω_j and equal to a_j for each $j = 1, 2$. Suppose that $a_2 < a_1$ and that Ω_1 is an open, bounded and strongly convex set with smooth boundary. Then we have a Carleman estimate like* (4) *for the Schrödinger equation* (1) *in the domain Ω.*

3. STABILITY OF THE INVERSE PROBLEM

As described in the introduction, will only give a local answer about the determination of the potential p. We will first work on a linearized version of the problem and consider the following Schrödinger equation :

$$\begin{cases} iu' + \text{ div}\,(a(x)\nabla u) + q(x)u = f(x)R(x,t), & \Omega \times (0,T) \\ u(x,t) = 0, & \partial\Omega \times (0,T) \\ u(x,0) = 0, & \Omega \end{cases} \tag{21}$$

Here we set $y = y(p)$ the weak solution to (1) and $u = u(f)$ the one to (21). If we formally linearize equation (1) around a non stationary solution, we obtain equation (21). In fact, we notice here that if we set $f = p - q$, $u = y(q) - y(p)$ and $R = y(p)$ on $\Omega \times (0,T)$, we obtain (21) after substraction of (1) with potential p from (1) with potential q and linearization.

Linear inverse problem : Is it possible to determine $f|_\Omega$ from the knowledge of the normal derivative $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega\times(0,T)}$ where R and p are given and u is the solution to (21)?

The following theorem proves that this inverse problem is well posed.

Theorem 10. *Let $q \in L^\infty(\Omega)$ and u be a solution of equation* (21)*. We assume that*

$$R \in W^{1,2}(0,T,L^\infty(\Omega)),$$

$$R(0) \textit{ is real valued and } |R(x,0)| \geq r_0 > 0, \;\; \textit{a.e. in } \overline{\Omega}.$$

There exists a constant $C = C(\Omega, T, \|q\|_{L^\infty(\Omega)}, R) > 0$ such that if

$$\frac{\partial u}{\partial \nu} \in H^1(0,T;L^2(\Gamma_0)),$$

then,

$$\|f\|_{L^2(\Omega)} \leq C \left\| a_2 \frac{\partial u}{\partial \nu} \right\|_{H^1(0,T;L^2(\partial\Omega))}. \tag{22}$$

Proof :
As we need to estimate $\dfrac{\partial u}{\partial \nu}$ in $H^1(0,T;L^2(\Gamma_0))$ norm, we work on the equation

satisfied by $v = u'$:

$$\begin{cases} iv' + \text{ div}\,(a(x)\nabla v) + q(x)v = f(x)R'(x,t), & \Omega \times (0,T) \\ v(x,t) = 0, & \partial\Omega \times (0,T) \\ v(x,0) = -if(x)R(x,0), & \Omega \end{cases} \tag{23}$$

The Carleman inequality we just obtained is the key of the proof. We extend the function v on $\Omega \times (-T,T)$ by the formula $v(x,t) = -\overline{v}(x,-t)$ for every $(x,t) \in \Omega \times (-T,0)$. Since $R(0)$ and f are real valued, $v \in C([-T,T]; H_0^1(\Omega))$ and $\dfrac{\partial v}{\partial \nu} \in L^2((-T,T) \times \Gamma)$. We also extend R on $\Omega \times (-T,T)$ by the formula $R(x,t) = \overline{R}(x,-t)$ for every $(x,t) \in \Omega \times (-T,0)$ and if we denote the extention of R' by the same notation, then $R' \in L^2(-T,T; W^{1,\infty}(\Omega))$. Thus, v satisfies the same equation (23), set in $(-T,T)$.

As defined in Theorem 3, for $k = 1,2$, we set $w^k = e^{-s\varphi^k} v$ and

$$P_1^{\psi^k} w^k = i\partial_t w^k + \text{ div}(a\nabla w^k) + s^2 a |\nabla \varphi^k|^2 w^k.$$

Therefore, we define the following:

$$I = \sum_{k=1}^{2} Im \int_{-T}^{0} \int_{\Omega} P_1^{\psi^k} w^k \, \overline{w^k} \, dxdt.$$

On the one hand,

$$\begin{aligned} I &= \sum_{k=1}^{2} Im \int_{-T}^{0} \int_{\Omega} P_1^{\psi^k} w^k \, \overline{w^k} \, dxdt \\ &= \sum_{k=1}^{2} Im \int_{-T}^{0} \int_{\Omega} \left(i\partial_t w^k + \text{ div}(a\nabla w^k) + s^2 a |\nabla \varphi^k|^2 w^k \right) \overline{w^k} \, dxdt \\ &= \sum_{k=1}^{2} \int_{-T}^{0} \int_{\Omega} Re \left(\partial_t w^k \overline{w^k} \right) - Im \left(a \left| \nabla w^k \right|^2 - s^2 a |\nabla \varphi^k|^2 \left| w^k \right|^2 \right) dxdt \\ &= \frac{1}{2} \sum_{k=1}^{2} \int_{-T}^{0} \int_{\Omega} \partial_t \left(|w^k|^2 \right) dxdt \\ &= \frac{1}{2} \sum_{k=1}^{2} \int_{\Omega} |w^k(x,0)|^2 \, dx \\ &= \frac{1}{2} \sum_{k=1}^{2} \int_{\Omega} |f(x)|^2 |R(x,0)|^2 e^{-2s\varphi^k(x,0)} \, dx. \end{aligned} \tag{24}$$

On the other hand, Cauchy-Schwarz inequality and Carleman estimate from Theorem 9 give :

$$\begin{aligned}
I &\leq \sum_{k=1}^{2} \left(\int_{-T}^{T} \int_{\Omega} |P_1^{\psi^k} w^k|^2 \, dxdt \right)^{\frac{1}{2}} \left(\int_{-T}^{T} \int_{\Omega} |w^k|^2 \, dxdt \right)^{\frac{1}{2}} \\
&\leq \sum_{k=1}^{2} \left\| P_1^{\psi^k}(w^k) \right\|_{L^2(Q)} \left\| w^k \right\|_{L^2(Q)} \\
&\leq Cs^{-\frac{3}{2}} \sum_{k=1}^{2} \left(\left\| P^{\psi^k}(w^k) \right\|_{L^2(Q)}^2 + s \iint_{\Sigma_+^{\psi^k}} \theta^k \left| a \frac{\partial w^k}{\partial \nu} \right|^2 \, d\sigma dt \right) \\
&\leq Cs^{-\frac{3}{2}} \sum_{k=1}^{2} \left(\iint_{Q} |fR'|^2 e^{-2s\varphi^k} \, dxdt + s \iint_{\Sigma_+^{\psi^k}} \theta^k \left| a_2 \frac{\partial v}{\partial \nu} \right|^2 e^{-2s\varphi^k} \, d\sigma dt \right).
\end{aligned}$$

Then, $\varphi^k(x,t) = \frac{\alpha - e^{\lambda\psi^k(x)}}{(T-t)(T+t)}$ is such that $e^{-2s\varphi^k(x,t)} \leq e^{-2s\varphi^k(x,0)}$ for all $x \in \Omega$ and $t \in (-T,T)$ and it is easy to see that $\theta e^{-2s\varphi}$ is bounded on $\Sigma_+^{\psi^k}$ and that using the definition of the extensions of v and R', we easily get

$$I \leq Cs^{-\frac{3}{2}} \sum_{k=1}^{2} \left(\int_0^T \int_{\Omega} |fR'|^2 e^{-2s\varphi^k(0)} \, dxdt + s \iint_{\Sigma_+^{\psi^k}} \left| a_2 \frac{\partial v}{\partial \nu} \right|^2 \, d\sigma dt \right) \tag{25}$$

¿From $R \in W^{1,2}(0,T,L^\infty(\Omega))$ and $|R(x,0)| \geq r_0 > 0$ almost everywhere in $\overline{\Omega}$, we deduce that

$$\exists\, g_0 \in L^2(0,T),\ |R'(x,t)| \leq g_0(t)|R(x,0)|,\ \forall x \in \Omega,\ t \in (0,T).$$

Hence, from (24) and (25) we have :

$$\begin{aligned}
&\sum_{k=1}^{2} \int_{\Omega} |f|^2 |R(0)|^2 e^{-2s\varphi^k(0)} \, dx \\
&\leq Cs^{-\frac{3}{2}} \sum_{k=1}^{2} \left(\int_0^T \int_{\Omega} |fR'|^2 e^{-2s\varphi^k(0)} \, dxdt + s \iint_{\Sigma_+^{\psi^k}} \left| a_2 \frac{\partial v}{\partial \nu} \right|^2 \, d\sigma dt \right) \\
&\leq C \sum_{k=1}^{2} s^{-\frac{3}{2}} \int_0^T \int_{\Omega} |f|^2 |g_0|^2 |R(0)|^2 e^{-2s\varphi^k(0)} \, dxdt \\
&\quad + C \sum_{k=1}^{2} s^{-\frac{1}{2}} \iint_{\Sigma_+^{\psi^k}} \left| a_2 \frac{\partial v}{\partial \nu} \right|^2 \, d\sigma dt.
\end{aligned}$$

But $g_0 \in L^2(0,T)$ implies $\displaystyle\int_0^T |g_0(t)|^2 \, dt \leq K < +\infty$ and so we can write

$$\left(1 - \frac{CK}{s^{\frac{3}{2}}}\right) \sum_{k=1}^{2} \int_{\Omega} |f|^2 |R(0)|^2 e^{-2s\varphi^k(0)} \, dx \leq Cs^{-\frac{1}{2}} \sum_{k=1}^{2} \iint_{\Sigma_+^{\psi^k}} \left| a_2 \frac{\partial v}{\partial \nu} \right|^2 \, d\sigma dt$$

that becomes easily, if s is large enough ($s > (CK)^{\frac{2}{3}}$) and C remains a generic positive constant

$$\int_\Omega |f|^2 |R(0)|^2 \left(e^{-2s\varphi^1(0)} + e^{-2s\varphi^2(0)}\right) dx \leq C s^{-\frac{1}{2}} \iint_\Sigma \left|a_2 \frac{\partial v}{\partial \nu}\right|^2 d\sigma dt.$$

Moreover, since $|R(x,0)| \geq r_0 > 0$ and $e^{-2s\varphi^k(x,0)} \geq e^{-2s\frac{\alpha-1}{T^2}} > 0$ almost everywhere in $\overline{\Omega}$, we obtain

$$\int_\Omega |f(x)|^2 \, dx \leq C \iint_\Sigma \left|a_2 \frac{\partial v}{\partial \nu}\right|^2 d\sigma dt,$$

and therefore, Theorem 10 has been proved. ■

Remark : if we replace the assumption "$R(0)$ *is real valued*" by the following "$R(0)$ *takes its values in* $i\mathbb{R}$", then the appropriate extensions for (x,t) in $\Omega \times (-T, 0)$ are $v(x,t) = \overline{v}(x,-t)$ and $R(x,t) = -\overline{R}(x,-t)$.

We will end this paper by the proof of Theorem 1 which is a direct consequence of Theorem 10. Indeed, if we set $\tilde{u} = y(q) - y(p)$, $f = p - q$ and $R = y(p)$, then $\tilde{u}$ is the solution of

$$\begin{cases} \tilde{u}' + \text{div}(a\nabla\tilde{u}) + (p-f)\tilde{u} = f(x)R(x,t) & (0,T)\times\Omega \\ \tilde{u} = 0 & (0,T)\times\Sigma \\ \tilde{u}(0) = 0 & \Omega \end{cases} \tag{26}$$

where $q = p - f \in \mathcal{U}$, with $\mathcal{U}$ bounded in $L^\infty(\Omega)$ from the hypothesis of Theorem 1. The key point is that in the proof of Theorem 10, all the constants $C > 0$ depend on the L^∞-norm of the potential. Thus, with $q \in \mathcal{U}$, we are actually, with equation (26) in a situation similar to the linear inverse problem related to equation (21) and we then obtain the desired result. ■

APPENDIX: DIRECT PROOF OF LEMMA 7

Without lost of generality, we can take $x_0 = 0$. Now, take $x, v \in \mathbb{R}^N \setminus \{0\}$ and define $g(t) = \mu^2(x + tv)$ for $t \in \mathbb{R}$. Then g depends only on the restriction of μ^2 to the plane $\Pi = \langle\{x, v\}\rangle \subset \mathbb{R}^N$ spanned by the vectors x and v. Moreover, by definition of ρ, it is not difficult to see that $\rho|_\Pi = \rho_0$, where we have denoted by ρ_0 the function defined in the plane Π as in (15), but where the closed curve is given by $\Gamma_1 = \Pi \cap \partial\Omega_1$, wich by hypothesis is strongly convex (see Remark 1).

It is not difficult to see that $\dfrac{d^2g}{dt^2}(0) = D^2(\mu^2)(x)(v,v)$ and then this expression depends only on the curve $\Gamma_1 \subset \Pi$. We conclude that it suffices to consider the two-dimensional case.

Assuming $N = 2$, Γ_1 can be parameterized in polar coordinates by

$$\gamma(\theta) = (\rho(\theta)\cos\theta, \rho(\theta)\sin\theta) \quad \theta \in [0, 2\pi).$$

The expression for the Hessian matrix of second derivatives in polar coordinates is

$$D^2(\mu^2) = Q_\theta H(\mu^2) Q_\theta^T$$

where Q_θ is the rotation matrix by angle θ, and

$$H(\mu^2) = \begin{pmatrix} \frac{\partial^2 \mu^2}{\partial r^2} & \frac{1}{r}\left(\frac{\partial^2 \mu^2}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \mu^2}{\partial \theta}\right) \\ \frac{1}{r}\left(\frac{\partial^2 \mu^2}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \mu^2}{\partial \theta}\right) & \frac{1}{r^2}\frac{\partial^2 \mu^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial \mu^2}{\partial r} \end{pmatrix}.$$

Now, since $x_0 = 0$, we have $\mu^2(\theta, r) = \dfrac{\bar{a}}{\rho(\theta)^2} r^2 + M$.

One can notice that μ^2 is well defined and smooth in $\Omega_0 \setminus B_\varepsilon(x_0)$ (which means $\{r \geq \varepsilon\} \setminus \Gamma_1$). All the computations that follows are valid in this set. We already said above that ρ is constant with respect to r and only depends on θ such that $\frac{\partial \rho}{\partial r} = 0$. Hence, we get

$$H(\mu^2) = \frac{2\bar{a}}{\rho^2} \begin{pmatrix} 1 & -\frac{\rho_\theta}{\rho} \\ -\frac{\rho_\theta}{\rho} & \frac{1}{\rho^2}(3\rho_\theta^2 - \rho\rho_{\theta\theta} + \rho^2) \end{pmatrix}, \tag{27}$$

where we have denoted $\rho_\theta = \frac{\partial \rho}{\partial \theta}$ and so on.

We will use the following well known lemma (see [12]) concerning curves in the plane:

Lemma 11. *Let γ be a C^2 curve in the plane parameterized in polar coordinates by its angle: $\gamma(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$. Then, the curvature of γ is given by the formula*

$$\kappa_\gamma(\theta) = \frac{r^2 + 2r_\theta^2 - r r_{\theta\theta}}{(r^2 + r_\theta^2)^{3/2}}.$$

Since the polar parametrization of Γ_1 is given by $r(\theta) = \rho(\theta)$ and Ω_1 is strongly convex, we obtain

$$\kappa_{\Gamma_1}(\theta) = \frac{\rho^2 + 2\rho_\theta^2 - \rho\rho_{\theta\theta}}{(\rho^2 + \rho_\theta^2)^{3/2}} > 0 \qquad \forall \theta \in [0, 2\pi[.$$

We will now prove that $H(\mu^2)$ is uniformly positive definite in $\Omega_0 \setminus B_\varepsilon(x_0)$, which will imply that the hypothesis (H_4) is fulfilled in this set, since a is piecewise constant.

The eigenvalues of the matrix $\frac{\rho^2}{2\bar{a}} H(\mu^2)$ satisfy the equation

$$r = \frac{1}{2}\left(d \pm \sqrt{d^2 - 4m}\right)$$

where $d = \dfrac{1}{\rho^2}\left(3\rho_\theta^2 - \rho\rho_{\theta\theta} + 2\rho^2\right)$ and

$$m = \frac{1}{\rho^2}\left(2\rho_\theta^2 - \rho\rho_{\theta\theta} + \rho^2\right) = \frac{(\rho^2 + \rho_\theta^2)^{3/2}}{\rho^2}\kappa_{\Gamma_1}(\theta) > 0.$$

Then $r_2 = \frac{1}{2}\left(d + \sqrt{d^2 - 4m}\right) \leq d$, and $r_1 = \frac{m}{r_2} \geq \frac{m}{d}$ for all $\theta \in [0, 2\pi)$. Since Ω_1 is bounded we get the desired result. ■

REFERENCES

[1] L. BAUDOUIN, A. MERCADO and A. OSSES, *A global Carleman estimate in a transmission wave equation and application to a one-measurement inverse problem*, Inverse Problems 23 (2007), no. 1, 257-278.

[2] L. BAUDOUIN and J.-P. PUEL, *Uniqueness and stability in an inverse problem for the Schrdinger equation*, Inverse Problems 18 (2002), no. 6, 1537-1554; Corrigendum in Inverse Problems 23 (2007), no. 3, 1327–1328.

[3] M. BELLASSOUED, *Uniqueness and stability in determining the speed of propagation of second-order hyperbolic equation with variable coefficients*. Appl. Anal. 83 (2004), no. 10, 983-1014.

[4] M. BELLASSOUED, M. YAMAMOTO, *Inverse source problem for a transmission problem for a parabolic equation* J. Inverse Ill-Posed Probl. 14 (2006), no. 1, 47-56.

[5] A. BENABDALLAH, Y. DERMENJIAN and J. LE ROUSSEAU, *Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient - and application to controllability and an inverse problem*, J. Math. Anal. Appl., 2007 (336), 2, p. 865-887.

[6] A. BENABDALLAH, P. GAITAN and J. LE ROUSSEAU, *Stability of discontinuous diffusion coefficients and initial conditions in an inverse problem for the heat equation*, http://hal.archives-ouvertes.fr/hal-00016490/fr

[7] A. L. BUKHGEIM, *Introduction to the theory of Inverse Problems*, Inverse and Ill-posed problem Series. VSP, Utrecht, 2000.

[8] A. L. BUKHGEIM and M. V. KLIBANOV, *Global uniqueness of a class of inverse problems*, Dokl. Akad. Nauk SSSR 260 (1981), no. 2, 269-272, English translation: Soviet Math. Dokl, 24 (2), (1982), 244-247.

[9] L. CARDOULIS, M. CRISTOFOL and P. GAITAN, *Inverse problem for the Schrödinger operator in an unbounded strip*. http://arxiv.org/abs/math/0612539v2

[10] G. CRASTA, A. MALUSA, *The distance function from the boundary in a Minkowski space*. Trans. Amer. Math. Soc. 359 (2007), no. 12, 5725–5759.

[11] A. DOUBOVA; A. OSSES and J.-P. PUEL, *Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients*, A tribute to J. L. Lions. ESAIM Control Optim. Calc. Var. 8 (2002), 621-661

[12] A. GRAY, *Modern differential geometry of curves and surfaces with Mathematica*. Second edition. CRC Press, Boca Raton, FL, 1998.

[13] O. Yu. IMANUVILOV and M. YAMAMOTO, *Determination of a coefficient in an acoustic equation with a single measurement*, Inverse Problems 19 (2003), no. 1, 157-171.

[14] O. Yu IMANUVILOV and M. YAMAMOTO, *Global uniqueness and stability in determining coefficients of wave equations*, Comm. Partial Differential Equations 26 (2001), no. 7-8, 1409-1425.

[15] V. ISAKOV, *Inverse Problems for Partial Differencial Equations*, Springer Verlag, Berlin, 1998.

[16] M. V. KLIBANOV and J. MALINSKY, *Newton-Kantorovich method for three-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with time-dependent data*, Inverse Problems, 7 (1991), 577-595.

[17] M.V. KLIBANOV, *Inverse Problems and Carleman estimates*, Inverse Problems **8** (1992), 575-596.

[18] I. LASIECKA, R. TRIGGIANI, P.-F. YAO, *Inverse/observability estimates for second-order hyperbolic equations with variable coefficients*, J. Math. Anal. Appl. 235 (1999), no. 1, 13-57.

[19] I. LASIECKA, R. TRIGGIANI and X. ZHANG, *Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. Part I: H1-estimates*, J. Inv. Ill-Posed Problems, Vol. 11(1) (2004) 43–123.

[20] E. MACHTYNGIER, *Exact controllability for the Schrödinger equation*, SIAM J. Control and Optimization **32** No 1 (1994), 24-34.

[21] A. MERCADO, A. OSSES and L. ROSIER, *Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights*, Inverse Problems 24 (2008) 015017.

[22] J.-P. PUEL and M. YAMAMOTO, *Generic well posedness in a multidimensional hyperbolic inverse problem*, J. of Inverse and ill-posed problems **1** (1997), 53-83.

[23] J.-P. PUEL and M. YAMAMOTO, *On a global estimate in a linear inverse hyperbolic problem*, Inverse Problems 12 (1996), no. 6, 995-1002.

[24] R. SCHNEIDER, *Convex bodies: the Brunn–Minkowski theory*, Cambridge Univ. Press, Cambridge, 1993.

[25] M. YAMAMOTO, *Uniqueness and stability in multidimensional hyperbolic inverse problems*, J. Math. Pures Appl. **78** (1999), 65-98.

Current address: Université de Toulouse, LAAS - CNRS, 7 avenue du Colonel Roche, F-31077 Toulouse, France.

E-mail address: baudouin@laas.fr

Current address: Department of Mathematics, University of Washington. C-446 Padelford Hall Box 354350. Seattle, Washington 98195-4350 USA. Partially supported by grant ECOS C04E08.

E-mail address: albertom@math.washington.edu