On the control of the linear Kuramoto-Sivashinsky equation*

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Abstract

In this paper we study the null controllability property of the linear Kuramoto-Sivashinsky equation by means of either boundary or internal controls. In the Dirichlet boundary case, we use the moment theory to prove that with only one boundary control, the null controllability property holds if and only if the anti-diffusion parameter of the equation does not belong to a critical set of numbers. Regarding the Neumann boundary case, we prove that the null controllability property does not hold with only one boundary control. However, it does always hold when two boundary controls or one internal control are considered. The proof of the latter is based on the controllability-observability duality and a suitable Carleman estimate.

Keywords: Kuramoto-Sivashinky equation, parabolic equation, boundary control, internal control, null controllability, moment theory, Carleman estimates.

1 Introduction

The Kuramoto-Sivashinsky equation is given by

$$z_t + z_{xxxx} + \lambda z_{xx} + zz_x = 0, \tag{1}$$

where $\lambda > 0$ is known as the anti-diffusion parameter. This equation was derived independently by Kuramoto and Tsuzuki [11] as a model for phase turbulence in reaction-diffusion systems, and by Sivashinsky [16], as a model for the physical phenomenon of plane flame propagation.

In this paper we study the control properties of the linear Kuramoto-Sivashinsky equation, that is to say equation (1) without the nonlinear term zz_x, by means of either boundary or internal controls. The type of boundary conditions studied are called the Dirichlet case and the Neumann case. Being more precise, we first consider boundary control inputs u_1, u_2 and study the following boundary control systems.

- **Dirichlet Case - Boundary Control.**

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = 0, & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = u_1(t),\ z(t,L) = 0, & t \in (0,T), \\ z_x(t,0) = u_2(t),\ z_x(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{2}$$

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- **Neumann Case - Boundary Control.**

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = 0, & (t,x) \in (0,T)\times(0,L), \\ z_{xx}(t,0) = u_1(t),\ z_{xx}(t,L) = 0, & t \in (0,T), \\ z_{xxx}(t,0) = u_2(t),\ z_{xxx}(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{3}$$

Given the parabolic character of these equations, the appropriate control notion to study is the null controllability, which is defined as follows. System (2) (respectively, system (3)) is said to be null controllable in time $T > 0$ if given any initial state z_0, there exist controls (u_1, u_2) such that the corresponding solution $z = z(t,x)$ of equation (2) (respectively, equation (3)) satisfies $z(T,\cdot) = 0$.

In the literature, there are already some control results for these systems. In [13], the stability and stabilizability issues have been studied for (2) and (3) by using four control inputs acting on the four boundary data. Again using four boundary controls, the robust control has been addressed in [10] for (2). The question of using less controls in (2) has been raised in [2] where the case $u_1 = 0$ is considered. By using the input u_2 only, the null controllability of (2) is proven to hold in [2] if and only if the anti-diffusion parameter does not belong to a set of critical values:

$$\lambda \notin \left\{ \frac{(j^2+k^2)\pi^2}{L^2} \ /\ (j,k) \in \mathbb{N}^2 \text{ with the same parity and } j < k \right\}. \tag{4}$$

In this paper, we complete the study in [2] by proving the following.

Theorem 1.1 *Consider $u_2 = 0$ and suppose that*

$$\lambda \notin \left\{ \frac{(j^2+k^2)\pi^2}{L^2} \ /\ (j,k) \in \mathbb{N}^2 \text{ with the same parity and } j < k \right\} \cup \left\{ \frac{4l^2\pi^2}{L^2} \ /\ l \in \mathbb{N} \right\}. \tag{5}$$

Then, for every $z_0 \in L^2(0,L)$ there exists $u_1 \in H^1(0,T)$ such that the unique solution $z \in C([0,T];L^2(0,L))$ of equation (2) satisfies $z(T,\cdot) = 0$ in $L^2(0,L)$.

The proof of Theorem 1.1 makes use of the fact that the operator

$$A_D : y \in H^4 \cap H_0^2(0,L) \subset L^2(0,L) \to -y'''' - \lambda y'' \in L^2(0,L), \tag{6}$$

which is the underlying spatial operator in equation (2), has a compact resolvent and is self-adjoint. Therefore, it has a discrete spectrum consisting only in real eigenvalues, $\{\sigma_k\}_{k\in\mathbb{N}}$, and its corresponding eigenfunctions, $\{\phi_k\}_{k\in\mathbb{N}}$, forms an orthonormal basis of $L^2(0,L)$. This allows us to transform the null controllability problem into a problem of moments (see Lemma 2.5), which is solved by using the moment theory developed by Fattorini and Russel [7], and the results on the asymptotic behaviour of σ_k and $\phi_k'''(0)$ when $k \to +\infty$ in [2].

Therefore, in the Dirichlet boundary case, the null controllability property holds with only one control provided that $\lambda > 0$, the anti-diffusion coefficient, does not belong to the corresponding critical set (5). The situation improves with two controls, where $\lambda > 0$ does not play any role in the null controllability property. This property holds with two controls and has already been shown in [2] and [3] by using the moment theory and a suitable Carleman estimate, respectively.

Regarding the Neumann case (3), up to our best knowledge, there are no controllability results. In this case, the corresponding underlying spatial operator is

$$A_N : y \in \{v \in H^4(0,L) \ /\ v'' \in H_0^2(0,L)\} \subset L^2(0,L) \to -y'''' - \lambda y'' \in L^2(0,L),$$

which has a compact resolvent but is not self-adjoint because of the boundary conditions. With a useful characterization of this property (see Lemma 3.5) we have obtained the following noncontrollability results with one control only.

Theorem 1.2 *(a) Consider $u_2 = 0$ and suppose that $z_0 \in L^2(0, L)$ is such that*

$$\int_0^L z_0(x) \cos(\sqrt{\lambda} x)\, dx \neq 0. \tag{7}$$

Then, for every $u_1 \in L^2(0, T)$ the unique solution $z \in C([0, T]; L^2(0, L))$ of equation (3) satisfies $z(T, \cdot) \neq 0$ in $L^2(0, L)$.

(b) Consider $u_1 = 0$ and suppose that $z_0 \in L^2(0, L)$ is such that

$$\int_0^L z_0(x) \sin(\sqrt{\lambda} x)\, dx \neq 0. \tag{8}$$

Then, for every $u_2 \in L^2(0, T)$ the unique solution $z \in C([0, T]; L^2(0, L))$ of equation (3) satisfies $z(T, \cdot) \neq 0$ in $L^2(0, L)$.

Once again, the situation improves with two controls and we are able to prove the following.

Theorem 1.3 *For every $z_0 \in L^2(0, L)$ there exist $(u_1, u_2) \in L^2(0, T)^2$ such that the unique solution $z \in C([0, T]; L^2(0, L))$ of equation (3) satisfies $z(T, \cdot) = 0$ in $L^2(0, L)$.*

The proof of this theorem uses the following internal control problem, where $\omega \subset [0, L]$ is a given nonempty open set and $\mathbb{1}_\omega$ stands for the characteristic function on ω.

- **Neumann Case - Internal Control.**

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = u \mathbb{1}_\omega, & (t, x) \in (0, T) \times (0, L), \\ z_{xx}(t, 0) = 0,\ z_{xx}(t, L) = 0, & t \in (0, T), \\ z_{xxx}(t, 0) = 0,\ z_{xxx}(t, L) = 0, & t \in (0, T), \\ z(0, x) = z_0(x), & x \in (0, L). \end{cases} \tag{9}$$

In this context, we obtain the next internal null controllability result.

Theorem 1.4 *For every $z_0 \in L^2(0, L)$ there exists $u \in L^2(0, T; L^2(\omega))$ such that the unique solution $z \in C([0, T]; L^2(0, L))$ of equation (9) satisfies $z(T, \cdot) = 0$ in $L^2(0, L)$.*

This is done with a Carleman estimates approach in order to prove an observability inequality. Finally, we make use of the regularizing effect of equation (3) when $u_1 = u_2 = 0$ (see Proposition 3.3) to prove that this internal control result implies Theorem 1.3. A similar internal control result has been obtained in the Dirichlet case in [18].

This paper is organized as follows. The Dirichlet case is studied in Section 2 by presenting well posedness results (Section 2.1), characterization of the null controllability (Section 2.2) and the proof of Theorem 1.1, (Section 2.3). The Neumann case is considered in Section 3 by presenting well posedness results (Section 3.1), the proof of the non-controllability result Theorem 1.2 (Section 3.2) and the proof of the control results Theorem 1.3 and Theorem 1.4. In Section 3.4, a suitable Carleman estimate is obtained.

2 Dirichlet Actuation

2.1 Well-Posedness

In this section we present the well-posedness results needed for studying the control system with Dirichlet boundary conditions. Let us consider the equation

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = f, & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = u_1(t),\ z(t,L) = 0, & t \in (0,T), \\ z_x(t,0) = u_2(t),\ z_x(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{10}$$

From [2], [3], [10] and [13] it is known that A_D, which is defined in (6), is an infinitesimal generator of a strongly continuous semigroup in $L^2(0,L)$. Therefore, from [14, Theorem 2.9, Chapter 4], it follows that the equation

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} = g, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = 0,\ y(t,L) = 0, & t \in (0,T), \\ y_x(t,0) = 0,\ y_x(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{11}$$

has a unique solution $y \in C([0,T]; H^4 \cap H_0^2(0,L)) \cap C^1([0,T]; L^2(0,L))$ provided that we have $g \in C^1([0,T]; L^2(0,L))$ and $y_0 \in H^4 \cap H_0^2(0,L)$. The above facts, the density of $H^4 \cap H_0^2(0,L)$ in $L^2(0,L)$ and $H_0^2(0,L)$, and suitable energy estimates, that can be obtained, for instance, by employing the techniques used in [1], [9] and [10], lead us to the following result.

Proposition 2.1 *Let $g \in L^2(0,T; L^2(0,L))$.*

(a) If $y_0 \in L^2(0,L)$, then equation (11) has a unique solution $y \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^2(0,L))$. Moreover, there exists $C = C(T,L,\lambda) > 0$ such that

$$\|y\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H_0^2(0,L))} \leq C \left(\|g\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{L^2(0,L)} \right).$$

(b) If $y_0 \in H_0^2(0,L)$, then equation (11) has a unique solution $y \in C([0,T]; H_0^2(0,L)) \cap L^2(0,T; H^4(0,L))$. Moreover, there exists $C = C(T,L,\lambda) > 0$ such that

$$\|y\|_{C([0,T];H_0^2(0,L))\cap L^2(0,T;H^4(0,L))} \leq C \left(\|g\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{H_0^2(0,L)} \right).$$

This proposition allows us to incorporate the boundary conditions to equation (10).

Proposition 2.2 *Let $f \in L^2(0,T; L^2(0,L))$, $(u_1,u_2) \in H^1(0,T)^2$ and $z_0 \in L^2(0,L)$. Then, equation (10) has a unique solution $z \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2(0,L))$. Moreover, there exists $C = C(T,L,\lambda) > 0$ such that*

$$\|z\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))} \leq C \left(\|f\|_{L^2(0,T;L^2(0,L))} + \|(u_1,u_2)\|_{H^1(0,T)^2} + \|z_0\|_{L^2(0,L)} \right). \tag{12}$$

Proof. By using the polynomials

$$d_1(x) := 2L^{-3}x^3 - 3L^{-2}x^2 + 1, \quad d_2(x) := L^{-2}x^3 - 2L^{-1}x^2 + x,$$

we define the auxiliary function

$$\psi_D(t,x) := u_1(t)d_1(x) + u_2(t)d_2(x).$$

By taking into account that $g := f - (\psi_D)_t - (\psi_D)_{xxxx} - \lambda(\psi_D)_{xx}$ and $y_0(x) := z_0(x) - \psi_D(0,x)$ are elements of $L^2(0,T;L^2(0,L))$ and $L^2(0,L)$, respectively, it follows that equation (11) has a unique solution $y \in C([0,T];L^2(0,L)) \cap L^2(0,T;H_0^2(0,L))$ thanks to Proposition 2.1 (a). Moreover, this solution satisfies

$$\|y\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H_0^2(0,L))} \leq C\left(\|g\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{L^2(0,L)}\right). \tag{13}$$

From

$$\psi_D(t,0) = u_1(t), \quad (\psi_D)_x(t,0) = u_2(t),$$

we get that $z := y + \psi_D \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))$ is a solution of equation (10). The continuous injection $H^1(0,T) \hookrightarrow C([0,T])$ allow us to get

$$\|\psi_D\|_{C([0,T];L^2(0,1))} \leq C\|(u_1,u_2)\|_{H^1(0,T)^2},$$

which combined with $\|z\| - \|\psi_D\| \leq \|y\|$ (valid for any norm) and (13) give us (12). This inequality and the linearity of the equation yield the uniqueness of solutions. The proof of Proposition 2.2 is complete. ■

We finish this section with a result concerning the regularizing effect of equation (10) when $u_1 = u_2 = 0$ and $f = 0$. This will be used in Section 3.1.

Proposition 2.3 *Let $\tau \in (0,T)$ and $z_0 \in L^2(0,L)$. Then, the unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;H_0^2(0,L))$ of equation (10) with $u_1 = u_2 = 0$ and $f = 0$, belongs to*

$$\mathcal{RD}(\tau,L) := C([\tau,T];H_0^2(0,L)) \cap L^2(\tau,T;H^4(0,L)) \cap H^1(\tau,T;L^2(0,L)).$$

Moreover, there exists $C = C(T,L,\lambda) > 0$ such that

$$\|z\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H_0^2(0,L))} + \|z\|_{\mathcal{RD}(\tau,L)} \leq C\left(1+\frac{1}{\tau}\right)^{1/2}\|z_0\|_{L^2(0,L)}. \tag{14}$$

Proof. We make all the computations needed considering $z_0 \in H^4 \cap H_0^2(0,L)$, so that equation (10) has a unique solution $z \in C([0,T];H^4 \cap H_0^2(0,L)) \cap C^1([0,T];L^2(0,L))$. The case $z_0 \in L^2(0,L)$ follows from a density argument after obtaining (14).

Let us consider $\phi = \phi(t)$ a smooth function and multiply equation (10) by $z_{xxxx}\phi$ to get

$$\frac{1}{2}\int_0^L \frac{\partial\left(|z_{xx}(t,x)|^2\right)}{\partial t}\phi(t)\,dx + \int_0^L |z_{xxxx}|^2\phi(t)\,dx + \int_0^L \lambda z_{xx}(t,x)z_{xxxx}(t,x)\phi(t)\,dx = 0. \tag{15}$$

Let $\tau \in (0,T)$ and $t \in [\tau,T]$. First, integrating (15) over (t,T) with $\phi(t) = 1$, and then using the Cauchy-Schwarz inequality, we obtain

$$\|z_{xx}(t,\cdot)\|_{L^2(0,L)}^2 \leq \|z_{xx}(T,\cdot)\|_{L^2(0,L)}^2 + 3\|z_{xxxx}\|_{L^2(\tau,T;L^2(0,L))}^2 + \lambda^2\|z_{xx}\|_{L^2(\tau,T;L^2(0,L))}^2. \tag{16}$$

Second, integrating by parts (15) over $(0,T)$ with $\phi(t) = t$, and then using the Cauchy-Schwarz inequality, we prove that

$$T\|z_{xx}(T,\cdot)\|_{L^2(0,L)}^2 + \int_0^T\int_0^L |z_{xxxx}|^2 t\,dxdt \leq (1+T\lambda^2)\|z_{xx}\|_{L^2(0,T;L^2(0,L))}^2,$$

from where we obtain

$$\|z_{xx}(T,\cdot)\|^2_{L^2(0,L)} + \|z_{xxxx}\|^2_{L^2(\tau,T;L^2(0,L))} \le \frac{C}{\tau}\|z_{xx}\|^2_{L^2(0,T;L^2(0,L))}.$$

Third, from the combination of (12) with the previous inequality and (16) we arrive to

$$\|z\|^2_{C([\tau,T];H^2_0(0,L))\cap L^2(\tau,T;H^4(0,L))} \le C\left(1+\frac{1}{\tau}\right)\|z_0\|^2_{L^2(0,L)}.$$

Note that here we have used the fact that $\|v^{(n)}\|_{L^2(0,L)}+\|v\|_{L^2(0,L)}$, with $n\in\mathbb{N}$, is a norm equivalent to $\|v\|_{H^n(0,1)}$. Moreover, the equation $z_t = -z_{xxxx} - \lambda z_{xx}$ tells us that we actually have

$$\|z\|^2_{C([\tau,T];H^2_0(0,L))\cap L^2(\tau,T;H^4(0,L))\cap H^1(\tau,T;L^2(0,L))} \le C\left(1+\frac{1}{\tau}\right)\|z_0\|^2_{L^2(0,L)}.$$

Finally, (12) together with this inequality yields (14). The proof of Proposition 2.3 is complete.
■

2.2 Boundary Control

In this section we develop useful characterizations of the null controllability property for the control system (2).

Lemma 2.4 *Let $T>0$ and consider $u_2=0$. Equation (2) is null controllable in time T in $L^2(0,L)$ if and only if for any $z_0\in L^2(0,L)$ there exists $u_1\in H^1(0,T)$ such that for every $q_T\in H^2_0(0,L)$ it holds*

$$\int_0^L z_0(x)q(0,x)\,dx = \int_0^T u_1(t)q_{xxx}(t,0)\,dt, \tag{17}$$

where $q=q(t,x)$ is the unique solution of the adjoint equation

$$\begin{cases} -q_t + q_{xxxx} + \lambda q_{xx} = 0, & (t,x)\in(0,T)\times(0,L),\\ q(t,0)=0,\ q(t,L)=0, & t\in(0,T),\\ q_x(t,0)=0,\ q_x(t,L)=0, & t\in(0,T),\\ q(T,x)=q_T(x), & x\in(0,L). \end{cases} \tag{18}$$

Proof. We have that equation (2) has a unique solution $z\in C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))$ and the adjoint equation (18) has a unique solution $q\in C([0,T];H^2_0(0,L))\cap L^2(0,T;H^4(0,L))$ thank to Propositions 2.2 and 2.1 (b) respectively.

Multiplying by q the equation (2) and then performing some integration by parts we get

$$\int_0^L z(T,x)q_T(x)\,dx - \int_0^L z_0(x)q(0,x)\,dx = -\int_0^T u_1(t)q_{xxx}(t,0)\,dt. \tag{19}$$

Note that $z(t,\cdot)\in L^2(0,L)$ for every $t\in[0,T]$ and $q_{xxx}(\cdot,0)\in L^2(0,T)$. On one hand, if $z(T,\cdot)=0$ in $L^2(0,L)$, then from (19) it follows (17). On the other hand, if (17) hold, then from (19) we obtain $(z_T,q_T)_{L^2(0,L)}=0$ for every $q_T\in H^2_0(0,L)$, leading us to $z(T,\cdot)=0$ in $L^2(0,L)$ since $H^2_0(0,L)$ is dense in $L^2(0,L)$. The proof of Lemma 2.4 is complete. ■

We shall make use of the following fact, taken from [2], to transform the null controllability problem into a problem of moments.

(F) A_D, defined in (6), is a self-adjoint operator whose resolvent is compact. Its spectrum is a discrete set consisting only of real eigenvalues, denoted by $\{\sigma_k\}_{k\in\mathbb{N}}$, satisfying $\sigma_k \leq \lambda^2/4$ for all $k \in \mathbb{N}$ and $\lim_{k\to+\infty} \sigma_k = -\infty$. Its corresponding eigenfunctions, denoted by $\{\phi_k\}_{k\in\mathbb{N}}$, are elements of $H^4 \cap H_0^2(0, L)$ and form an orthonormal basis of $L^2(0, L)$.

Lemma 2.5 *Let $T > 0$ and consider $u_2 = 0$. Equation (2) is null controllable in time T in $L^2(0, L)$ if and only if for any*

$$z_0(x) = \sum_{k\in\mathbb{N}} z_0^k \phi_k(x), \quad \sum_{k\in\mathbb{N}} |z_0^k|^2 < +\infty, \tag{20}$$

there exists $f \in H^1(0, T)$ such that

$$\phi_k'''(0) \int_0^T f(t) e^{\sigma_k t}\, dt = z_0^k e^{\sigma_k T}, \quad \forall k \in \mathbb{N}. \tag{21}$$

The control is given by $u_1(t) := f(T - t)$.

Proof. Assume that equation (2) is null controllable in time T in $L^2(0, L)$. Hence, Lemma 2.4 in particular tells us that (17) holds for $q_T = \phi_k \in H^4 \cap H_0^2(0, L)$ for every $k \in \mathbb{N}$. Since the corresponding solution of equation (18) is $q(t, x) = e^{(T-t)\sigma_k} \phi_k(x)$, we arrive to (21) with $f(t) = u_1(T - t)$ by considering (20) and the orthonormality of $\{\phi_k\}_{k\in\mathbb{N}}$ in $L^2(0, L)$.

Now assume that (21) holds. We know that every $q_T \in H_0^2(0, L)$ can be written as

$$q_T(x) = \sum_{k\in\mathbb{N}} q_T^k \phi_k(x), \quad \sum_{k\in\mathbb{N}} |q_T^k|^2 < +\infty.$$

After some computations, we find that the corresponding unique solution of equation (18), that is an element of $C([0, T]; H_0^2(0, L)) \cap L^2(0, T; H^4(0, L))$ in virtue of Proposition 2.1 (b), is

$$q(t, x) = \sum_{k\in\mathbb{N}} q_T^k e^{(T-t)\sigma_k} \phi_k(x). \tag{22}$$

Multiplying by q_T^k to (21), then considering the orthonormality of $\{\phi_k\}_{k\in\mathbb{N}}$ in $L^2(0, L)$ and finally adding from $k = 1$ to $k = n \in \mathbb{N}$, lead us to

$$\int_0^L \left(\sum_{k\in\mathbb{N}} z_0^k \phi_k(x) \right) \left(\sum_{k=1}^n q_T^k e^{\sigma_k T} \phi_k(x) \right) dx = \int_0^T f(T - t) \left(\sum_{k=1}^n q_T^k e^{(T-t)\sigma_k} \phi_k'''(0) \right) dt. \tag{23}$$

Thank to the regularity of (22) we can let $n \to +\infty$ in (23) to obtain (17) with $u_1(t) = f(T - t)$. Therefore, Lemma 2.4 gives us the desired result. The proof of Lemma 2.5 is complete. ■

2.3 Spectral Analysis

The purpose of this section is to prove Theorem 1.1, which is one of our main results. The proof is based on the spectral analysis of the operator A_D, which is defined in (6).

Proof of Theorem 1.1. Every initial state $z_0 \in L^2(0, L)$ can be written as

$$z_0(x) = \sum_{k\in\mathbb{N}} z_0^k \phi_k(x), \quad \sum_{k\in\mathbb{N}} |z_0^k|^2 < +\infty.$$

In virtue of Lemma 2.5, the control system (2) will be null controllable in time T in $L^2(0,L)$ if and only if there exists $f \in H^1(0,T)$ such that

$$\phi_k'''(0) \int_0^T f(t) e^{\sigma_k t}\, dt = z_0^k e^{\sigma_k T}, \quad \forall k \in \mathbb{N}.$$

Hence, provided that $\phi_k'''(0) \neq 0$ for every $k \in \mathbb{N}$, we arrive to the following problem of moments. Find $f \in H^1(0,T)$ such that

$$\int_0^T f(t) e^{\sigma_k t}\, dt = \frac{z_0^k e^{\sigma_k T}}{\phi_k'''(0)}, \quad \forall k \in \mathbb{N}.$$

This problem of moments can directly be solved by using the moment theory developed by Fattorini and Russel (Corollary 3.2 in [7]), and the results on the asymptotic behaviour of σ_k and $\phi_k'''(0)$ when $k \to +\infty$ in [2, Lemma 2.2]. Therefore, the only thing left to prove is to determine when $\phi_k'''(0) \neq 0$ for every $k \in \mathbb{N}$.

In order to make the notation clearer, we omit the subscript k for the eigenvalues and eigenfunctions. Let (ϕ, σ) satisfy

$$\begin{cases} -\phi'''' - \lambda \phi'' = \sigma \phi, & x \in (0,L), \\ \phi(0) = \phi(L) = \phi'(0) = \phi'(L) = 0. \end{cases} \tag{24}$$

We consider the following three cases: $\sigma > 0$, $\sigma = 0$ or $\sigma < 0$. In the sake of completeness, we write again all the computations of the eigenfunctions, even if they appear in [2].

Case 1: $\sigma > 0$. Because of **(F)** we know that there exist a finite number of positive eigenvalues satisfying $\sigma \leq \lambda^2/4$. Set $\alpha := 2^{-1/2}\left(\lambda - \sqrt{\lambda^2 - 4\sigma}\right)^{1/2}$ and $\beta := 2^{-1/2}\left(\lambda + \sqrt{\lambda^2 - 4\sigma}\right)^{1/2}$. We have that

$$\phi(x) = C_1 \cos(\alpha(x - L/2)) + C_2 \sin(\alpha(x - L/2)) + C_3 \cos(\beta(x - L/2)) + C_4 \sin(\beta(x - L/2)),$$

is a solution of equation (24). Here, the real constants C_1, C_2, C_3 and C_4 are solutions of

$$\underbrace{\begin{bmatrix} \cos(\alpha L/2) & \cos(\beta L/2) \\ \alpha \sin(\alpha L/2) & \beta \sin(\beta L/2) \end{bmatrix}}_{S_1} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{25}$$

$$\underbrace{\begin{bmatrix} \sin(\alpha L/2) & \sin(\beta L/2) \\ \alpha \cos(\alpha L/2) & \beta \cos(\beta L/2) \end{bmatrix}}_{S_2} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{26}$$

From (25) and (26) we get two finite sets of positive eigenvalues, which we denote by $\{\sigma_{1,n}\}_{n=1}^{m_1}$ and $\{\sigma_{2,n}\}_{n=1}^{m_2}$ for a certain $(m_1, m_2) \in \mathbb{N}^2$.

(a) $\{\sigma_{1,n}\}_{n=1}^{m_1}$ is obtained from the positive solutions of $\det(S_1) = 0$. Thus, they satisfy

$$\alpha \sin(\alpha L/2) \cos(\beta L/2) = \beta \sin(\beta L/2) \cos(\alpha L/2). \tag{27}$$

The following two possibilities are considered. The first possibility is when $\cos(\alpha L/2) \neq 0$. From (25), (26) and (27) we get that the eigenfunctions associated to $\sigma_{1,n}$ are

$$\phi_A(x) = C_3\left[-\frac{\cos(\beta L/2)}{\cos(\alpha L/2)}\cos(\alpha(x-L/2)) + \cos(\beta(x-L/2))\right],$$
$$\phi_B(x) = C_4\left[-\frac{\beta\cos(\beta L/2)}{\alpha\cos(\alpha L/2)}\sin(\alpha(x-L/2)) + \sin(\beta(x-L/2))\right].$$

Considering (27) we arrive to

$$\phi_A'''(0) = C_3\beta(\alpha^2-\beta^2)\sin(\beta L/2), \quad \phi_B'''(0) = C_4\beta(\alpha^2-\beta^2)\cos(\beta L/2).$$

Let us study when $\phi_A'''(0) \neq 0$ and $\phi_B'''(0) \neq 0$. On the one hand, if $\phi_A'''(0) = 0$, then from (27) we get $\sin(\alpha L/2) = 0$, allowing is to conclude, together with $\beta^2 - \alpha^2 = \sqrt{\lambda^2 - 4\sigma}$ and $\beta^2 + \alpha^2 = \lambda$, that $\lambda > 0$ should be of the form $\lambda = \left[(2j)^2 + (2k)^2\right]\pi^2/L^2$ with $(j,k) \in \mathbb{N}^2$ being such that $j < k$. On the other hand, from (27) we conclude that $\phi_B'''(0) \neq 0$.

The second possibility is when $\cos(\alpha L/2) = 0$. From (25), (26) and (27) we get that the eigenfunctions associated to $\sigma_{1,n}$ are

$$\phi_A(x) = C_3\left[-\frac{\beta\sin(\beta L/2)}{\alpha\sin(\alpha L/2)}\cos(\alpha(x-L/2)) + \cos(\beta(x-L/2))\right],$$
$$\phi_B(x) = C_4\left[-\frac{\sin(\beta L/2)}{\sin(\alpha L/2)}\sin(\alpha(x-L/2)) + \sin(\beta(x-L/2))\right],$$

Considering (27) we arrive to

$$\phi_A'''(0) = C_3\beta(\alpha^2-\beta^2)\sin(\beta L/2), \quad \phi_B'''(0) = C_4\beta(\alpha^2-\beta^2)\cos(\beta L/2).$$

As we did before, let us study when $\phi_A'''(0) \neq 0$ and $\phi_B'''(0) \neq 0$. On the one hand, from (27) we conclude that $\phi_A'''(0) \neq 0$. On the other hand, from (27) we get $\phi_B'''(0) = 0$, allowing is to conclude, together with $\beta^2 - \alpha^2 = \sqrt{\lambda^2 - 4\sigma}$ and $\beta^2 + \alpha^2 = \lambda$, that $\lambda > 0$ should be of the form $\lambda = \left[(2j+1)^2 + (2k+1)^2\right]\pi^2/L^2$ with $(j,k) \in \mathbb{N}^2$ being such that $j < k$.

(b) $\{\sigma_{2,n}\}_{n=1}^{m_2}$ is obtained from the positive solutions of $\det(S_2) = 0$. Thus, they satisfy

$$\alpha\sin(\beta L/2)\cos(\alpha L/2) = \beta\sin(\alpha L/2)\cos(\beta L/2).$$

This time we consider the possibilities $\sin(\alpha L/2) \neq 0$ and $\sin(\alpha L/2) = 0$, where the computations and conclusions are the same as those obtained in the possibilities $\cos(\alpha L/2) = 0$ and $\cos(\alpha L/2) \neq 0$, respectively, of the previous part.

Therefore, if $\lambda \notin \left\{\frac{(j^2+k^2)\pi^2}{L^2} \,/\, (j,k) \in \mathbb{N}^2 \text{ with the same parity and } j < k\right\}$, then $\phi'''(0) \neq 0$.

Case 2: $\sigma = 0$. We have that

$$\phi(x) = C_1 + C_2(x - L/2) + C_3\cos\left(\sqrt{\lambda}(x-L/2)\right) + C_4\sin\left(\sqrt{\lambda}(x-L/2)\right),$$

is a solution of equation (24). Here, the real constants C_1, C_2, C_3 and C_4 are solutions of

$$\underbrace{\begin{bmatrix} 1 & \cos(\sqrt{\lambda}L/2) \\ 0 & \sqrt{\lambda}\sin(\sqrt{\lambda}L/2) \end{bmatrix}}_{S_1}\begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{28}$$

$$\underbrace{\begin{bmatrix} L & 2\sin(\sqrt{\lambda}L/2) \\ 1 & \sqrt{\lambda}\cos(\sqrt{\lambda}L/2) \end{bmatrix}}_{S_2} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{29}$$

The following two possibilities are considered.

(a) Let us assume that $\det(S_1) = \sqrt{\lambda}\sin(\sqrt{\lambda}L/2) = 0$, which is when $\sqrt{\lambda}L/2 = l\pi$ with $l \in \mathbb{N}$. Since $\det(S_2) = L\sqrt{\lambda}\cos(\sqrt{\lambda}L/2) - 2\sin(\sqrt{\lambda}L/2) \neq 0$, we get that $C_2 = C_4 = 0$ is the unique solution of system (29), allowing us to conclude that $\phi'''(0) = -C_3\lambda\sqrt{\lambda}\sin(\sqrt{\lambda}L/2) = 0$. Note that $\lambda \neq 4l^2\pi^2/L^2$ with $l \in \mathbb{N}$ gives us $\phi'''(0) = -C_4\lambda\sqrt{\lambda}\cos(\sqrt{\lambda}L/2) \neq 0$ thank to $\det(S_1) \neq 0$.

(b) Let us assume that $\det(S_2) = L\sqrt{\lambda}\cos(\sqrt{\lambda}L/2) - 2\sin(\sqrt{\lambda}L/2) = 0$, which tells us that $\det(S_1) \neq 0$. Since the unique solution of system (28) is $C_1 = C_3 = 0$, we get that $\phi'''(0) = -C_4\lambda\sqrt{\lambda}\cos(\sqrt{\lambda}L/2) \neq 0$.

Therefore, if $\lambda \notin \left\{\dfrac{4l^2\pi^2}{L^2} \,/\, l \in \mathbb{N}\right\}$, then $\phi'''(0) \neq 0$.

<u>Case 3: $\sigma < 0$.</u> Set $\alpha := 2^{-1/2}\left(-\lambda + \sqrt{\lambda^2 - 4\sigma}\right)^{1/2}$ and $\beta := 2^{-1/2}\left(\lambda + \sqrt{\lambda - 4\sigma}\right)^{1/2}$. We have that

$$\phi(x) = C_1\cosh(\alpha(x - L/2)) + C_2\sinh(\alpha(x - L/2)) + C_3\cos(\beta(x - L/2)) + C_4\sin(\beta(x - L/2)),$$

is a solution of equation (24). Here, the real constants C_1, C_2, C_3 and C_4 are solutions of

$$\underbrace{\begin{bmatrix} \cosh(\alpha L/2) & \cos(\beta L/2) \\ \alpha\sinh(\alpha L/2) & \beta\sin(\beta L/2) \end{bmatrix}}_{S_1} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{30}$$

$$\underbrace{\begin{bmatrix} \sinh(\alpha L/2) & \sin(\beta L/2) \\ \alpha\cosh(\alpha L/2) & \beta\cos(\beta L/2) \end{bmatrix}}_{S_2} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{31}$$

From (30) and (31) we get two sets of negative eigenvalues, which we denote by $\{\sigma_{1,n}\}_{n\in\mathbb{N}}$ and $\{\sigma_{2,n}\}_{n\in\mathbb{N}}$.

(a) $\{\sigma_{1,n}\}_{n\in\mathbb{N}}$ is obtained from the negative solutions of $\det(S_1) = 0$. Thus, they satisfy

$$-\alpha\sinh(\alpha L/2)\cos(\beta L/2) = \beta\sin(\beta L/2)\cosh(\alpha L/2). \tag{32}$$

Considering (32) we get

$$\begin{aligned} \det(S_2) &= \beta\cos(\beta L/2)\sinh(\alpha L/2) - \alpha\sin(\beta L/2)\cosh(\alpha L/2), \\ &= -(\alpha^2 + \beta^2)\sin(\beta L/2)\cosh(\alpha L/2)/\alpha, \end{aligned}$$

which tells us that $C_2 = C_4 = 0$ is the unique solution of system (31). Accordingly, from (30) we get that the eigenfunction associated to $\sigma_{1,n}$ is

$$\phi(x) = C_3\left[-\frac{\beta\sin(\beta L/2)}{\alpha\sin(\alpha L/2)}\cos(\alpha(x - L/2)) + \cos(\beta(x - L/2))\right].$$

Once again, (32) allows us to conclude that

$$\phi'''(0) = -C_3\beta(\alpha^2+\beta^2)\sin(\beta L/2) \neq 0.$$

(b) $\{\sigma_{2,n}\}_{n\in\mathbb{N}}$ is obtained from the negative solutions of $\det(S_2)=0$. Thus, they satisfy

$$\alpha\sin(\beta L/2)\cosh(\alpha L/2) = \beta\cos(\beta L/2)\sinh(\alpha L/2). \tag{33}$$

Considering (33) we get

$$\begin{aligned}\det(S_2) &= -\beta\sin(\beta L/2)\cosh(\alpha L/2) - \alpha\cos(\beta L/2)\sinh(\alpha L/2),\\ &= -(\alpha^2+\beta^2)\cos(\beta L/2)\sinh(\alpha L/2)/\alpha.\end{aligned}$$

which tells us that $C_1 = C_3 = 0$ is the unique solution of system (30). Accordingly, from (31) we get that the eigenfunction associated to $\sigma_{2,n}$ is

$$\phi(x) = C_4\left[-\frac{\sin(\beta L/2)}{\sinh(\alpha L/2)}\sinh(\alpha(x-L/2)) + \sin(\beta(x-L/2))\right].$$

Once again, (33) allows us to conclude that

$$\phi'''(0) = -C_4\beta(\alpha^2+\beta^2)\cos(\beta L/2) \neq 0.$$

Therefore, for every $\lambda > 0$ we have that $\phi'''(0) \neq 0$.

From the combination of the above three cases we conclude that if

$$\lambda \notin \left\{\frac{(j^2+k^2)\pi^2}{L^2} \;/\; (j,k)\in\mathbb{N}^2 \text{ with the same parity and } j<k\right\} \cup \left\{\frac{4\pi^2 l^2}{L^2} \;/\; l\in\mathbb{N}\right\},$$

then $\phi_k'''(0) \neq 0$ for every $k\in\mathbb{N}$. The proof of Theorem 1.1 is complete. ∎

Remark 2.6 *Suppose that $\phi_k'''(0) = 0$ for some $k\in\mathbb{N}$. Lemma 2.5 tells us that the initial state $\phi_k \in H^4\cap H_0^2(0,L)$ for equation (2) can not be driven to the null state in time T.*

3 Neumann Actuation

3.1 Well-Posedness

In this section we present the well-posedness results needed for studying the control system with Neumann boundary conditions. Let us consider the equation

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = f, & (t,x)\in(0,T)\times(0,L),\\ z_{xx}(t,0) = u_1(t),\ z_{xx}(t,L) = 0, & t\in(0,T),\\ z_{xxx}(t,0) = u_2(t),\ z_{xxx}(t,L) = 0, & t\in(0,T),\\ z(0,x) = z_0(x), & x\in(0,L),\end{cases} \tag{34}$$

We begin by studying this equation with homogeneous boundary conditions.

Proposition 3.1 *Let* $f \in C^1([0,T]; L^2(0,L))$ *and* $z_0 \in \mathcal{N}_L := \{v \in H^4(0,L) \,/\, v'' \in H_0^2(0,L)\}$. *Then, equation (34) with* $u_1 = u_2 = 0$ *has a unique solution* $z \in C([0,T]; \mathcal{N}_L) \cap C^1([0,T]; L^2(0,L))$.

Proof. We use the semigroup theory for the proof of this proposition. Let us consider the bilinear form $a : H^2(0,L) \times H^2(0,L) \to \mathbb{R}$ defined by

$$a(u,v) := \int_0^L u''(x)v''(x)\,dx + \int_0^L \lambda u''(x)v(x)\,dx.$$

Let A_N be the unbounded operator with domain

$$D(A_N) := \{u \in H^2(0,L) \,/\, v \mapsto a(u,v) \text{ is continuous over } H^2(0,L) \text{ for the topology of } L^2(0,L)\},$$

defined through $(A_N u, v)_{L^2(0,L)} = a(u,v)$. From the continuous injection $H^m(0,L) \hookrightarrow C^{m-1}([0,L])$ for $m \in \mathbb{N}$, the identity

$$a(u,v) = (u'''' + \lambda u'', v)_{L^2(0,L)} + u''(x)v'(x)\big|_{x=0}^{x=L} - u'''(x)v(x)\big|_{x=0}^{x=L},$$

and the Cauchy-Schwarz inequality, we get that $D(A_N) = \mathcal{N}_L$ and $A_N : D(A_N) \subset L^2(0,1) \to L^2(0,L)$ is given by $A_N u = u'''' + \lambda u''$.

It turns out that A_N is the underlying spatial operator of equation (34). Therefore, [14, Theorem 2.9, Chapter 4] would allow us to conclude our result if $-A_N$ is an infinitesimal generator of a strongly continuous semigroup in $L^2(0,L)$. The inequality

$$\begin{aligned}\int_0^L |v''(x)|^2 dx + \frac{1}{2}\int_0^L |v(x)|^2 dx &= \frac{1}{2}\int_0^L |v(x)|^2 dx - \int_0^L \lambda v''(x)v(x)\,dx + a(v,v),\\ &\le \frac{1+\lambda^2}{2}\int_0^L |v(x)|^2 dx + \frac{1}{2}\int_0^L |v''(x)|^2 dx + a(v,v),\end{aligned}$$

and the fact that $\|v''\|_{L^2(0,L)} + \|v\|_{L^2(0,L)}$ is an equivalent norm to the norm $\|v\|_{H^2(0,L)}$, lead us to the existence of a $\lambda_0 \in \mathbb{R}$ and an $\alpha > 0$ such that for every $v \in H^2(0,L)$ it holds

$$\alpha \|v\|_{H^2(0,L)} \le \lambda_0 \|v\|^2_{L^2(0,L)} + a(v,v).$$

Accordingly, [6, Theorem 3, Chapter XVII] gives us the desired property for $-A_N$. The proof of Proposition 3.1 is complete. ■

This proposition allows us to incorporate nonhomogeneous boundary conditions to equation (34).

Proposition 3.2 *Let* $f \in L^2(0,T; L^2(0,L))$, $(u_1,u_2) \in L^2(0,T)^2$ *and* $z_0 \in L^2(0,L)$. *Then, equation (34) has a unique solution* $z \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2(0,L))$. *Moreover, there exists* $C = C(T,L,\lambda) > 0$ *such that*

$$\|z\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))} \le C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|(u_1,u_2)\|_{L^2(0,T)^2} + \|z_0\|_{L^2(0,L)}\right). \quad (35)$$

Proof. Let us assume that $f \in C^1([0,T]; L^2(0,L))$, $u_1, u_2 \in \{u \in C^2([0,T]) \,/\, u(0) = 0\}$ and $z_0 \in \mathcal{N}_L$. With the aid of the polynomials

$$n_1(x) := (1/10)L^{-3}x^5 - (1/4)L^{-2}x^4 + (1/2)x^2, \quad n_2(x) := (1/20)L^{-2}x^5 - (1/6)L^{-1}x^4 + (1/6)x^3,$$

we define the auxiliary function

$$\psi_N(t,x) := u_1(t)n_1(x) + u_2(t)n_2(x).$$

By taking into account that $g := f-(\psi_N)_t-(\psi_N)_{xxxx}-\lambda(\psi_N)_{xx}$ is an element of $C^1([0,T];L^2(0,L))$, it follows that the equation

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} = g, & (t,x) \in (0,T)\times(0,L), \\ y_{xx}(t,0) = 0,\ y_{xx}(t,L) = 0, & t\in(0,T), \\ y_{xxx}(t,0) = 0,\ y_{xxx}(t,L) = 0, & t\in(0,T), \\ y(0,x) = z_0(x), & x\in(0,L), \end{cases}$$

has a unique solution $y \in C([0,T];\mathcal{N}_L)\cap C^1([0,T];L^2(0,L))$ in virtue of Proposition 3.1. From

$$(\psi_N)_{xx}(t,0) = u_1(t), \quad (\psi_N)_{xxx}(t,0) = u_2(t),$$

we get that $z := y+\psi_N \in C([0,T];H^4(0,L))\cap C^1([0,T];L^2(0,L))$ is a solution of equation (34). We would like to have a unique solution in $C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))$ to this equation. In order to achieve this, we shall derive a suitable energy estimate and use the linearity of the equation.

Multiplying by z to equation (34) we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |z(t,x)|^2dx\right) + \int_0^L z_{xxxx}(t,x)z(t,x)\,dx + \int_0^L \lambda z_{xx}(t,x)z(t,x)\,dx = \int_0^L f(t,x)z(t,x)\,dx. \tag{36}$$

Two integrations by parts and the Cauchy-Schwarz inequality lead us to

$$\begin{aligned}\int_0^L z_{xxxx}(t,x)z(t,x)\,dx &= \int_0^L |z_{xx}(t,x)|^2dx - z_{xx}(t,x)z_x(t,x)\big|_{x=0}^L + z_{xxx}(t,x)z(t,x)\big|_{x=0}^L, \\ &\geq \int_0^L |z_{xx}(t,x)|^2dx - \frac{1}{2\varepsilon}\left(|u_1(t)|^2+|u_2(t)|^2\right) - \varepsilon\|z(t,\cdot)\|^2_{W^{1,\infty}(0,L)},\end{aligned}$$

where $\varepsilon > 0$ will be chosen later. Combining the above inequality with (36) and then using the continuous injection $H^2(0,L)\hookrightarrow W^{1,\infty}(0,L)$ we obtain

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L |z(t,x)|^2dx\right) + \int_0^L |z_{xx}(t,x)|^2dx \leq \int_0^L |f(t,x)|^2dx + (1+\lambda^2)\int_0^L |z(t,x)|^2dx \\ + \frac{1}{2\varepsilon}\left(|u_1(t)|^2+|u_2(t)|^2\right) + C\varepsilon\|z(t,\cdot)\|^2_{H^2(0,L)}.\end{aligned} \tag{37}$$

Applying Grönwall's Lemma to (37) gives us

$$\begin{aligned}\|z\|^2_{C([0,T];L^2(0,L))} + \|z\|^2_{L^2(0,T;L^2(0,L))} \leq C\Big(&\|f\|^2_{L^2(0,T;L^2(0,L))} + \varepsilon^{-1}\|(u_1,u_2)\|^2_{L^2(0,T)^2} \\ &+\|z_0\|^2_{L^2(0,L)} + \varepsilon\|z\|^2_{L^2(0,T;H^2(0,L))}\Big).\end{aligned}$$

By integrating (37) over $(0,T)$ and then combining it with the last inequality we get

$$\begin{aligned}\|z\|^2_{C([0,T];L^2(0,L))} + \|z\|^2_{L^2(0,T;H^2(0,L))} \leq C\Big(&\|f\|^2_{L^2(0,T;L^2(0,L))} + \varepsilon^{-1}\|(u_1,u_2)\|^2_{L^2(0,T)^2} \\ &+\|z_0\|^2_{L^2(0,L)} + \varepsilon\|z\|^2_{L^2(0,T;H^2(0,L))}\Big).\end{aligned}$$

Therefore, we arrive to (35) after the choice of $\varepsilon = 1/(2C)$ in the previous inequality. Since $C^1([0,T];L^2(0,L))$, $\{u \in C^2([0,T]) \ / \ u(0) = 0\}$ and $\mathcal{N}_L$ are dense in $L^2(0,T;L^2(0,L))$, $L^2(0,T)$ and $L^2(0,L)$, respectively, (35) allows us to use a density argument to conclude that equation (34) has a unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))$ if $f \in L^2(0,T;L^2(0,L))$, $(u_1,u_2) \in L^2(0,T)^2$ and $z_0 \in L^2(0,L)$. The proof of Proposition 3.2 is complete. ■

Our next result concerns the regularizing effect of equation (34) when $u_1 = u_2 = 0$ and $f = 0$. This will play a key role in the proof of Theorem 1.3.

Proposition 3.3 *Let $\tau \in (0,T)$ and $z_0 \in L^2(0,L)$. Then, the unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))$ of equation (34) with $u_1 = u_2 = 0$ and $f = 0$ belongs to*

$$\mathcal{RN}(\tau,L) := C([\tau,T];\mathcal{N}_L) \cap L^2(\tau,T;H^6(0,L)) \cap H^1(\tau,T;H^2(0,L)).$$

Moreover, there exists $C > 0$, depending on T and λ, such that

$$\|z\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))} + \|z\|_{\mathcal{RN}(\tau,L)} \leq C\left(1+\frac{1}{\tau}\right)^{1/2} \|z_0\|_{L^2(0,L)}. \tag{38}$$

Proof. For a $\tau \in (0,T)$ given, consider $\tau_1 \in (0,\tau)$. By noting that we can carry out the same computations as those made in the proof of Proposition 2.3, it is possible to obtain

$$\|z\|_{C([\tau_1,T];H^2(0,L))\cap L^2(\tau_1,T;\mathcal{N}_L)\cap H^1(\tau_1,T;L^2(0,L))} \leq C\left(1+\frac{1}{\tau_1}\right)^{1/2} \|z_0\|_{L^2(0,L)},$$

where z satisfies (34) with $u_1 = u_2 = 0$ and $f = 0$. Since $y = z_{xx}$ satisfies the equation

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} = 0, & (t,x) \in (\tau_1,T)\times(0,L), \\ y(t,0) = 0, \ y(t,L) = 0, & t \in (\tau_1,T), \\ y_x(t,0) = 0, \ y_x(t,L) = 0, & t \in (\tau_1,T), \\ y(\tau_1,x) = z_{xx}(\tau_1,x), & x \in (0,L), \end{cases}$$

and $z_{xx}(\tau_1,x)$ belongs to $L^2(0,L)$, we can use Proposition 2.3 to conclude that

$$z_{xx} \in \mathcal{RD}(\tau,L) := C([\tau,T];H_0^2(0,L)) \cap L^2(\tau,T;H^4(0,L)) \cap H^1(\tau,T;L^2(0,L)),$$

which gives us our result. Note that (38) follows from the combination of (35) and (14). The proof of Proposition 3.3 is complete. ■

We finish this section by studying the well-posedness of

$$\begin{cases} -q_t + q_{xxxx} + \lambda q_{xx} = 0, & (t,x) \in (0,T)\times(0,L), \\ (q_{xx}+\lambda q)(t,0) = 0, \ (q_{xx}+\lambda q)(t,L) = 0, & t \in (0,T), \\ (q_{xxx}+\lambda q_x)(t,0) = 0, \ (q_{xxx}+\lambda q_x)(t,L) = 0, & t \in (0,T), \\ q(T,x) = q_T(x), & x \in (0,L), \end{cases} \tag{39}$$

which is the adjoint equation associated to (3).

Proposition 3.4 *(a) If $q_T \in \mathcal{N}_L^* := \{v \in H^4(0,L) \ / \ (v'' + \lambda v) \in H_0^2(0,L)\}$, then equation (39) has a unique solution $q \in C([0,T];\mathcal{N}_L^*) \cap C^1([0,T];L^2(0,L))$.*

(b) *If $q_T \in L^2(0,L)$, then equation (39) has a unique solution $q \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))$. Moreover, there exists $C = C(T,L,\lambda) > 0$ such that*

$$\|q\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))} \leq C\|q_T\|_{L^2(0,L)}.$$

Proof. Let us recall some notation introduced in Proposition 3.1. Consider the unbounded operator $A_N : \mathcal{N}_L \subset L^2(0,L) \to L^2(0,L)$ given by $A_N u = u'''' + \lambda u''$. In that proposition we have shown that $-A_N$ is an infinitesimal generator of a strongly continuous semigroup in $L^2(0,L)$.

Let $u \in \mathcal{N}_L$ and $v \in H^4(0,1)$. From

$$(A_N u, v)_{L^2(0,L)} = \big(u, v'''' + \lambda v''\big)_{L^2(0,L)} + u'(x)(v''(x) + \lambda v(x))\big|_{x=0}^{x=L} - u(x)(v'''(x) + \lambda v'(x))\big|_{x=0}^{x=L},$$

we find out that $D(A_N^*) = \mathcal{N}_L^*$ and $A_N^* v = v'''' + \lambda v''$, which allow us to conclude that A_N is not a self-adjoint operator. However, because of the above mentioned property for $-A_N$, [14, Corollary 10.6, Chapter 1] tells us that $(-A_N)^*$ is also an infinitesimal generator of a strongly continuous semigroup in $L^2(0,L)$.

Therefore, on the one hand, part (a) of this proposition follows from the application of [14, Theorem 2.9, Chapter 4] to equation (39) after the change of variable $t \to T - t$. On the other hand, part (b) of this proposition follows by using the same arguments as those used in the proof of Proposition 3.2 by taking into account that $\mathcal{N}^*$ is dense in $L^2(0,L)$. The proof of Proposition 3.4 is complete. ■

3.2 Boundary control with one input

The aim of this section is to prove Theorem 1.2.

Lemma 3.5 *Let $T > 0$. Equation (3) is null controllable in time T in $L^2(0,L)$ if and only if for any $z_0 \in L^2(0,L)$ there exists $(u_1,u_2) \in L^2(0,T)^2$ such that for every $q_T \in L^2(0,L)$ it holds*

$$\int_0^L z_0(x)q(0,x)\,dx = \int_0^T u_1(t)q_x(t,0)\,dt - \int_0^T u_2(t)q(t,0)\,dt, \tag{40}$$

where $q = q(t,x)$ is the unique solution of the adjoint equation (39).

Proof. We have that equation (3) and the adjoint equation (39) have a unique solution in $C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L))$ thanks to Propositions 3.2 and 3.4 (b), respectively.

Multiplying equation (3) by q and then performing some integration by parts, we get

$$\int_0^L z(T,x)q_T(x)\,dx - \int_0^L z_0(x)q(0,x)\,dx = -\int_0^T u_1(t)q_x(t,0)\,dt + \int_0^T u_3(t)q(t,0)\,dt. \tag{41}$$

Note that $z(t,\cdot) \in L^2(0,L)$ for every $t \in [0,T]$ and $q_x(\cdot,0) \in L^2(0,T)$. On one hand, if $z(T,\cdot) = 0$ in $L^2(0,L)$, then from (41) it follows (40). On the other hand, if (40) holds, then from (41) we obtain $(z_T, q_T)_{L^2(0,L)} = 0$ for every $q_T \in L^2(0,L)$, leading us to $z(T,\cdot) = 0$ in $L^2(0,L)$. The proof of Lemma 3.5 is complete. ■

Proof of Theorem 1.2. Let $u_2 = 0$ and consider $z_0 \in L^2(0,L)$, satisfying (7), as the initial state of equation (3). If we take $q_T(x) = \cos(\sqrt{\lambda}x)$ as the final state of equation (39), then we get that $q(t,x) = \cos(\sqrt{\lambda}x)$ is its unique solution. Hence, for every $u_1 \in L^2(0,T)$ we have that

- $\displaystyle\int_0^L z_0(x)q(0,x)\,dx \neq 0$, thanks to (7).
- $\displaystyle\int_0^T u_1(t)q_x(t,0)\,dt = 0$.

Therefore, the preceding points and (40) of Lemma 3.5 give us part (a) of this theorem. For the other part, take $u_1 = 0$ and consider $z_0 \in L^2(0,L)$, satisfying (8), as the initial state of equation (3). In this case, the final state $q_T(x) = \sin(\sqrt{\lambda}x)$ in equation (39) gives the unique solution $q(t,x) = \sin(\sqrt{\lambda}x)$. Hence, for every $u_2 \in L^2(0,T)$ we have that

- $\displaystyle\int_0^L z_0(x)q(0,x)\,dx \neq 0$, thanks to (8).
- $\displaystyle\int_0^T u_3(t)q(t,0)\,dt = 0$.

Accordingly, the preceding points together with (40) of Lemma 3.5 lead us to part (b) of this theorem. The proof of Theorem 1.2 is complete. ■

3.3 Boundary control with two inputs and internal control

This section is devoted to the proofs of Theorem 1.3 and Theorem 1.4.

Recall that ω is a domain being such that $\overline{\omega} \subset (0,L)$. Throughout this section and the next one we use $Q := (0,T)\times(0,L)$ and $Q_\omega := (0,T)\times\omega$. Due to the controllability-observability duality (e.g. [4, Theorem 2.44] or [17, Theorem 11.2.1]), Theorem 1.4 can be shown by means of the following observability inequality.

Proposition 3.6 *There exists $C = C(L,\lambda,\omega) > 0$ such that*

$$\|q(0,\cdot)\|^2_{L^2(0,L)} \leq \frac{C}{T-2\varepsilon}\exp\left\{C\left(T-\varepsilon+\frac{\max\{T,T^2\}}{\varepsilon^2}\right)\right\}\iint_{Q_\omega}|q|^2 dxdt, \tag{42}$$

where $\varepsilon \in \left(0,\frac{T}{2}\right)$ and $q = q(t,x)$ is the unique solution of the adjoint equation (39) with $q_T \in L^2(0,L)$.

In our case, the main tool for obtaining this observability inequality is a Carleman estimate for the adjoint equation (39), which will be derived by following a procedure due to A.V. Fursikov and O.Yu. Imanuvilov [8]. To this end, we need several results as well as the introduction of two weight functions.

Lemma 3.7 (Lemma 1.1 in [5]) *Let ω_0 be a domain such that $\overline{\omega_0} \subset (0,L)$. Then, there exists a function $\psi \in C^4([0,L])$ satisfying*

$$\psi(x) > 0 \ \ \forall x \in (0,L), \quad \psi(0) = \psi(L) = 0, \quad |\psi'(x)| > 0 \ \ \forall x \in \overline{(0,L)}\setminus\omega_0. \tag{43}$$

The reader may also consult [8, Lemma 1.1, Chapter 1] for the proof of this lemma. We consider ω_0 to be a fixed domain such that $\overline{\omega_0} \subset \omega$, then we take the corresponding function ψ given by Lemma 3.7 and for $\mu > 0$ we introduce the weight functions

$$\alpha(t,x) := \frac{e^{4\mu\|\psi\|_{L^\infty(0,L)}} - e^{\mu\left(2\|\psi\|_{L^\infty(0,L)}+\psi(x)\right)}}{t(T-t)}, \quad \beta(t,x) := \frac{e^{\mu\left(2\|\psi\|_{L^\infty(0,L)}+\psi(x)\right)}}{t(T-t)}, \quad \forall (t,x) \in Q. \tag{44}$$

The last result needed is a Carleman estimate for the adjoint equation (18). Hence, for $q = q(t,x)$ to be made precise, consider the operator $Lq := -q_t + q_{xxxx} + \lambda q_{xx}$.

Theorem 3.8 (Theorem 1.1 in [18]) *There exist* $\mu_0 \geq 1$ *and* $C = C(L, \lambda, \omega) > 0$ *such that for every* $\mu \geq \mu_0$ *and* $\nu \geq \max\{T, T^2\}$ *we have*

$$\iint_Q e^{-2\nu\alpha} \left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2 + \nu^3\mu^4\beta^3|q_{xx}|^2 + \nu\mu^2\beta|q_{xxx}|^2\right) dxdt$$
$$+\iint_Q e^{-2\nu\alpha}\left(\frac{|q_t|^2 + |q_{xxxx}|^2}{\nu\beta}\right) dxdt \leq C\iint_Q e^{-2\nu\alpha}|Lq|^2 dxdt + C\iint_{Q_{\omega_1}} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2 dxdt, \tag{45}$$

with ω_1 *being any domain satisfying* $\overline{\omega_0} \subset \omega_1 \subseteq \omega$ *and* $q = q(t, x)$ *the unique solution of the adjoint equation (18) with* $q_T \in H_0^2(0, L)$.

Now we can present our Carleman estimate.

Proposition 3.9 *There exist* $\mu_0 \geq 1$ *and* $C = C(L, \lambda, \omega) > 0$ *such that for every* $\mu \geq \mu_0$ *and* $\nu \geq \max\{T, T^2\}$ *we have*

$$\iint_Q e^{-2\nu\alpha} \left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2 + \nu^3\mu^4\beta^3|q_{xx}|^2 + \nu\mu^2\beta|q_{xxx}|^2\right) dxdt$$
$$+\iint_Q e^{-2\nu\alpha}\left(\frac{|q_t|^2 + |q_{xxxx}|^2}{\nu\beta}\right) dxdt \leq C\iint_Q e^{-2\nu\alpha}\left(\frac{1}{\mu^2}|(Lq)_{xx}|^2 + \nu^3\mu^2\beta^3|Lq|^2\right) dxdt$$
$$+ C\iint_{Q_{\omega_1}} e^{-2\nu\alpha}\nu^{11}\mu^{10}\beta^{11}|q|^2 dxdt, \tag{46}$$

with ω_1 *being any domain satisfying* $\overline{\omega_0} \subset \omega_1 \subseteq \omega$ *and* $q = q(t, x)$ *the unique solution of the adjoint equation (39) with* $q_T \in \mathcal{N}_L^* := \{v \in H^4(0, L) \ / \ (v'' + \lambda v) \in H_0^2(0, L)\}$.

The proof of this proposition will be given in Section 3.4.

Remark 3.10 *Under the hypotheses of Proposition 3.9, we have that* $(Lq)_{xx} \in L^2(0, T; L^2(0, L))$. *In fact, by setting* $z := q_{xx} + \lambda q$ *we see that* z *satisfies equation (18) with a final state in* $H_0^2(0, L)$. *Therefore, Proposition 2.1 (b) allow us to conclude the assertion by taking into account that* $Lz = (Lq)_{xx} + \lambda Lq$.

A density argument together with Proposition 3.4 (b) can be used to obtain from Proposition 3.9 the following result.

Corollary 3.11 *There exist* $\mu_0 \geq 1$ *and* $C = C(L, \lambda, \omega) > 0$ *such that for every* $\mu \geq \mu_0$ *and* $\nu \geq \max\{T, T^2\}$ *we have*

$$\iint_Q e^{-2\nu\alpha} \left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2 + \nu^3\mu^4\beta^3|q_{xx}|^2\right) dxdt \leq C\iint_{Q_{\omega_1}} e^{-2\nu\alpha}\nu^{11}\mu^{10}\beta^{11}|q|^2 dxdt, \tag{47}$$

with ω_1 *being any domain satisfying* $\overline{\omega_0} \subset \omega_1 \subseteq \omega$ *and* $q = q(t, x)$ *the unique solution of the adjoint equation (39) with* $q_T \in L^2(0, L)$.

In what follows we derive the required observability inequality (42) given by Proposition 3.6.

Proof of Proposition 3.6. First, let us assume that $q_T \in \mathcal{N}_L^*$, so that the adjoint equation (39) has a unique solution $q \in C([0, T]; \mathcal{N}_L^*) \cap C^1([0, T]; L^2(0, L))$ thanks to Proposition 3.4 (a). Once again, the $q_T \in L^2(0, L)$ case follows by density argument and Proposition 3.4 (b) after obtaining inequality (42).

For $t \in [0,T]$, we define $E(t) := \|q(t,\cdot)\|^2_{L^2(0,L)}$. Multiplying equation (39) by $q = q(t,x)$ and then using the Cauchy-Schwarz inequality we get

$$-\frac{dE(t)}{dt} \leq \lambda^2 E(t).$$

This inequality together with

$$\frac{d}{dt}\left(e^{\lambda^2 t}E(t)\right)e^{-\lambda^2 t} = \lambda^2 E(t) + \frac{dE(t)}{dt},$$

leads us to

$$\|q(0,\cdot)\|^2_{L^2(0,L)} \leq e^{\lambda^2 t}\|q(t,\cdot)\|^2_{L^2(0,L)}, \quad t \in [0,T]. \tag{48}$$

Second, in (47) of Corollary 3.11 we fix $\mu = \mu_0$ and $\nu = \max\{T, T^2\}$ to obtain

$$\iint_Q e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt \leq C\iint_{Q_\omega} e^{-2\nu\alpha}\nu^{11}\mu^{10}\beta^{11}|q|^2dxdt. \tag{49}$$

Let $\varepsilon \in \left(0, \frac{T}{2}\right)$. Since there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{C_1\nu^7}{T^{14}}\exp\{-\frac{C_2\nu}{\varepsilon^2}\} \leq e^{-2\alpha\nu}\nu^7\mu^8\beta^7, \quad \forall (t,x) \in (\varepsilon, T-\varepsilon)\times(0,L),$$

$$e^{-2\alpha\nu}\nu^{11}\mu^{10}\beta^{11} \leq C_2, \quad \forall (t,x) \in Q,$$

from (49) it follows that

$$\int_\varepsilon^{T-\varepsilon}\int_0^1 |q|^2dxdt \leq \frac{CT^{14}}{\nu^7}\exp\left\{\frac{C\nu}{\varepsilon^2}\right\}\iint_{Q_\omega}|q|^2dxdt.$$

Finally, the observability inequality (42) is obtained from the combination of (48) and the preceding inequality. The proof of Proposition 3.6 is complete. ■

Proof of Theorem 1.3. Let $z_0 \in L^2(0,L)$ and $\tau \in (0,T)$. First, in virtue of Proposition 3.3 it follows that the auxiliar equation

$$\begin{cases} a_t + a_{xxxx} + \lambda a_{xx} = 0, & (t,x) \in (0,T)\times(0,L), \\ a_{xx}(t,0) = 0,\ a_{xx}(t,L) = 0, & t \in (0,T), \\ a_{xxx}(t,0) = 0,\ a_{xxx}(t,L) = 0, & t \in (0,T), \\ a(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{50}$$

has a unique solution $a \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2(0,L))$ which belongs to

$$\mathcal{RN}(\tau, L) := C([\tau,T]; \mathcal{N}_L) \cap L^2(\tau,T; H^6(0,L)) \cap H^1(\tau,T; H^2(0,L)).$$

Second, let $u \in L^2(0, T-\tau; L^2(\omega))$ where ω is a domain being such that $\overline{\omega} \subset (L, 2L)$. By taking into account that $a(\tau,\cdot) \in \mathcal{N}_L$, we consider a second auxiliar equation, which is given by

$$\begin{cases} b_t + b_{xxxx} + \lambda b_{xx} = u\mathbb{1}_\omega, & (t,x) \in (0,T-\tau)\times(0,2L), \\ b_{xx}(t,0) = 0,\ b_{xx}(t,2L) = 0, & t \in (0,T-\tau), \\ b_{xxx}(t,0) = 0,\ b_{xxx}(t,2L) = 0, & t \in (0,T-\tau), \\ b(0,x) = \mathbb{1}_\omega(x)a(\tau,x), & x \in (0,2L). \end{cases} \tag{51}$$

Due to Proposition 3.2, equation (51) has a unique solution $b \in C([0, T-\tau]; L^2(0, 2L)) \cap L^2(0, T-\tau; H^2(0, 2L))$. Moreover, by employing the methods used in Section 3.1, we also have that $b \in L^2(0, T-\tau; \mathcal{N}_{2L})$. An application of Theorem 1.4 tells us that there exists $u \in L^2(0, T-\tau; L^2(\omega))$ such that the unique solution of equation (51) satisfies $b(T-\tau, \cdot) = 0$ in $L^2(0, 2L)$.

Third, by defining $c(t, x) := b(t-\tau, x)$, with $(t, x) \in (\tau, T) \times (0, L)$, we get that $c(T, \cdot) = 0$ in $L^2(0, L)$. Furthermore, thank to equation (51) it follows that $c = c(t, x)$ satisfies the equation

$$\begin{cases} c_t + c_{xxxx} + \lambda c_{xx} = 0, & (t,x) \in (\tau, T) \times (0, L), \\ c_{xx}(t, 0) = 0,\ c_{xx}(t, L) = a_{xx}(t, L), & t \in (\tau, T), \\ c_{xxx}(t, 0) = 0,\ c_{xxx}(t, L) = a_{xxx}(t, L), & t \in (\tau, T), \\ c(\tau, x) = a(\tau, x), & x \in (0, L). \end{cases} \tag{52}$$

Note that in virtue of the continuous injection $H^4(0, L) \hookrightarrow C^3([0, L])$ we have that $a_{xx}(\cdot, L)$ and $a_{xxx}(\cdot, L)$ are elements of $L^2(0, T)$.

Finally, let us define

$$z(t, x) := \begin{cases} a(t, x) & , \quad (t, x) \in (0, \tau) \times (0, L), \\ c(t, L-x) & , \quad (t, x) \in (\tau, T) \times (0, L). \end{cases}$$

From equations (50) and (52) we have that $z = z(t, x)$ satisfies the equation

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = 0, & (t,x) \in (0, T) \times (0, L), \\ z_{xx}(t, 0) = u_1(t),\ z_{xx}(t, L) = 0, & t \in (0, T), \\ z_{xxx}(t, 0) = u_2(t),\ z_{xxx}(t, L) = 0, & t \in (0, T), \\ z(0, x) = z_0(x), & x \in (0, L), \end{cases}$$

with u_1 and u_2 being elements of $L^2(0, T)$ that are defined by

$$u_1(t) := \begin{cases} 0 & , \quad t \in (0, \tau), \\ c_{xx}(t, L) & , \quad t \in (\tau, T). \end{cases} \qquad u_2(t) := \begin{cases} 0 & , \quad t \in (0, \tau), \\ -c_{xxx}(t, L) & , \quad t \in (\tau, T). \end{cases}$$

Therefore, Proposition 3.2 and the fact that $z(T, \cdot) = 0$ in $L^2(0, L)$ allow us to conclude our result. The proof of Theorem 1.3 is complete. ■

3.4 Carleman Estimate

In this section we prove Proposition 3.9. The following properties of the weight functions defined in (44), that can be deduced from straightforward computations, are listed in the following lemma.

Lemma 3.12 *For every $(t, x) \in Q$ it holds:*

(a) $\mu \leq \dfrac{T^2}{8\|\psi\|_{L^\infty(0,L)}} \beta(t, x)$.

(b) Let $n \in \mathbb{N}$. There exists $C > 0$ such that $\left| \dfrac{\partial^n \alpha}{\partial x^n}(t, x) \right| \leq C \sum_{i=1}^{n} \mu^i \beta(t, x)$.

Proof of Proposition 3.9. For $\nu > 0$ let us consider $Pw := e^{-\nu\alpha} L(e^{\nu\alpha} w)$, with $w := e^{-\nu\alpha} q$, and the decomposition $Pw = P_1 w + P_2 w + P_3 w$ given by

$$
\begin{aligned}
P_1 w &:= - w_t + 4\nu^3 \alpha_x^3 w_x + 4\nu \alpha_x w_{xxx}, \\
P_2 w &:= \nu^4 \alpha_x^4 w + 6\nu^2 \alpha_x^2 w_{xx} + w_{xxxx}, \\
P_3 w &:= - \nu \alpha_t w + 6\nu^3 \alpha_{xx} \alpha_x^2 w + 3\nu^2 \alpha_{xx}^2 w + 4\nu^2 \alpha_{xxx} \alpha_x w + \nu \alpha_{xxxx} w \\
&\qquad + 12\nu^2 \alpha_{xx} \alpha_x w_x + 4\nu \alpha_{xxx} w_x + 6\nu \alpha_{xx} w_{xx} + \lambda \left(\nu^2 \alpha_x^2 w + \nu \alpha_{xx} w + 2\nu \alpha_x w_x + w_{xx}\right).
\end{aligned}
$$

We remark that this decomposition for Pw is slightly different to the decompositions considered in Theorem 3.1 of [1] and in Proposition 3 of [9]. Regardless of the above, the structure of the decomposition gives us

$$
\|P_1 w\|_{L^2(Q)}^2 + 2\left(P_1 w, P_2 w\right)_{L^2(Q)} + \|P_2 w\|_{L^2(Q)}^2 = \|Pw - P_3 w\|_{L^2(Q)}^2. \tag{53}
$$

We shall deduce the desired Carleman estimate from this equality and Theorem 3.8. This will be done in seven steps.

Step 1. We proceed to handle the terms $\|P_1 w\|_{L^2(Q)}^2$ and $\|P_2 w\|_{L^2(Q)}^2$. To this end we introduce the key quantities

$$
\|w\|_A := \iint_Q \left(\nu^7 \mu^8 \beta^7 |w|^2 + \nu^5 \mu^6 \beta^5 |w_x|^2 + \nu^3 \mu^4 \beta^3 |w_{xx}|^2 + \nu \mu^2 \beta |w_{xxx}|^2\right) dxdt,
$$

$$
\|w\|_B := \iint_Q \left(\frac{|w_t|^2 + |w_{xxxx}|^2}{\nu \beta}\right) dxdt.
$$

From the weight functions defined in (44) we have that $\alpha_x(t,x) = -\mu \psi'(x) \beta(t,x)$ for every $(t,x) \in Q$. The inequalities

$$
\iint_Q \frac{|w_t|^2}{\nu \beta} dxdt \leq \iint_Q \frac{|P_1 w|^2}{\nu \beta} dxdt + C \iint_Q \left(\nu^5 \mu^6 \beta^5 |w_x|^2 + \nu \mu^2 \beta |w_{xxx}|^2\right) dxdt,
$$

$$
\iint_Q \frac{|w_{xxxx}|^2}{\nu \beta} dxdt \leq \iint_Q \frac{|P_2 w|^2}{\nu \beta} dxdt + C \iint_Q \left(\nu^7 \mu^8 \beta^7 |w|^2 + \nu^3 \mu^4 \beta^3 |w_{xx}|^2\right) dxdt,
$$

together with the fact that $\beta(t,x) \geq 4/T^2$ for every $(t,x) \in Q$ allow us to conclude that

$$
\|w\|_B \leq C \left(\|P_1 w\|_{L^2(Q)}^2 + \|P_2 w\|_{L^2(Q)}^2 + \|w\|_A\right), \quad \forall \nu \geq T^2. \tag{54}
$$

Step 2. We proceed to compute $(P_1 w, P_2 w)_{L^2(Q)}$. For $i, j = 1, 2, 3$ we denote by $I_{i,j}$ the L^2-product in Q between the i-th term of $P_1 w$ with the j-th term of $P_2 w$. Note that with this notation we have

$$
(P_1 w, P_2 w)_{L^2(Q)} = \sum_{i,j=1}^{3} I_{i,j}.
$$

Integrations by parts are performed and each resulting term is labeled. When performing these computations it is considered that $w \in C([0,T]; \mathcal{N}_L) \cap C^1([0,T]; L^2(0,L))$ satisfies $w(0,x) = w(T,x) = 0$ for every $x \in (0,L)$. The latter is due to $w = e^{-\nu \alpha} q$ and the choice of the weight functions defined in (44).

Each resulting expression for $I_{i,j}$ is listed below.

- $I_{1,1} = \underbrace{\frac{\nu^4}{2} \iint_Q \left(\alpha_x^4\right)_t |w|^2 dxdt}_{R(w)}.$

- $I_{1,2} = \underbrace{-6\nu^2 \iint_Q \alpha_x^2 w_t w_{xx}\, dxdt}_{R(w)}.$

- $I_{1,3} = \underbrace{-\iint_Q w_t w_{xxxx}\, dxdt}_{R(w)}.$

- $I_{2,1} = \underbrace{-2\nu^7 \iint_Q \left(\alpha_x^7\right)_x |w|^2 dxdt}_{M_0(w)} + \underbrace{2\nu^7 \int_0^T \alpha_x^7 |w|^2\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)}.$

- $I_{2,2} = \underbrace{-12\nu^5 \iint_Q \left(\alpha_x^5\right)_x |w_x|^2 dxdt}_{M_1(w)} + \underbrace{12\nu^5 \int_0^T \alpha_x^5 |w_x|^2\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)}.$

- $I_{2,3} = \underbrace{-2\nu^3 \iint_Q \left(\alpha_x^3\right)_{xxx} |w_x|^2 dxdt}_{R(w)} + \underbrace{6\nu^3 \iint_Q \left(\alpha_x^3\right)_x |w_{xx}|^2 dxdt}_{M_2(w)} + \underbrace{2\nu^3 \int_0^T \left(\alpha_x^3\right)_{xx} |w_x|^2\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)}$

  $\underbrace{-2\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)} \underbrace{-4\nu^3 \int_0^T \left(\alpha_x^3\right)_x w_x w_{xx}\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)} + \underbrace{4\nu^3 \int_0^T \alpha_x^3 w_x w_{xxx}\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)}.$

- $I_{3,1} = \underbrace{-2\nu^5 \iint_Q \left(\alpha_x^5\right)_{xxx} |w|^2 dxdt}_{R(w)} + \underbrace{6\nu^5 \iint_Q \left(\alpha_x^5\right)_x |w_x|^2 dxdt}_{M_1(w)} + \underbrace{2\nu^5 \int_0^T \left(\alpha_x^5\right)_{xx} |w|^2\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)}$

  $\underbrace{-2\nu^5 \int_0^T \alpha_x^5 |w_x|^2\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)} \underbrace{-4\nu^5 \int_0^T \left(\alpha_x^5\right)_x w w_x\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)} + \underbrace{4\nu^5 \int_0^T \alpha_x^5 w w_{xx}\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)}.$

- $I_{3,2} = \underbrace{-12\nu^3 \iint_Q \left(\alpha_x^3\right)_x |w_{xx}|^2 dxdt}_{M_2(w)} + \underbrace{12\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)}.$

- $I_{3,3} = \underbrace{-2\nu \iint_Q \alpha_{xx} |w_{xxx}|^2 dxdt}_{M_3(w)} + \underbrace{2\nu \int_0^T \alpha_x |w_{xxx}|^2\big|_{x=0}^{x=L}\, dt}_{B(w,\cdot)}.$

Accordingly, by adding all the above terms we get

$$(P_1 w, P_2 w)_{L^2(Q_T)} = \sum_{k=0}^{3} M_k(w) + B(w,1) - B(w,0) + R(w), \tag{55}$$

where we have defined the quantities

$$M_0(w) := -14\nu^7 \iint_{Q_T} \alpha_x^6 \alpha_{xx} |w|^2 dxdt, \tag{56}$$

$$M_1(w) := -30\nu^5 \iint_{Q_T} \alpha_x^4 \alpha_{xx} |w_x|^2 dxdt, \tag{57}$$

$$M_2(w) := -18\nu^3 \iint_{Q_T} \alpha_x^2 \alpha_{xx} |w_{xx}|^2 dxdt, \tag{58}$$

$$M_3(w) := -2\nu \iint_{Q_T} \alpha_{xx} |w_{xxx}|^2 dxdt, \tag{59}$$

as the main terms,

$$\begin{aligned} B(w,x) := & \, 2\nu^7 \int_0^T \alpha_x^7 |w|^2 dt + 12\nu^5 \int_0^T \alpha_x^5 |w_x|^2 dt + 2\nu^3 \int_0^T \left(\alpha_x^3\right)_{xx} |w_x|^2 dt \\ & - 2\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2 dt - 4\nu^3 \int_0^T \left(\alpha_x^3\right)_x w_x w_{xx} dt + 4\nu^3 \int_0^T \alpha_x^3 w_x w_{xxx} dt \\ & + 2\nu^5 \int_0^T \left(\alpha_x^5\right)_{xx} |w|^2 dt - 2\nu^5 \int_0^T \alpha_x^5 |w_x|^2 dt - 4\nu^5 \int_0^T \left(\alpha_x^5\right)_x w w_x dt \\ & + 4\nu^5 \int_0^T \alpha_x^5 w w_{xx} dt + 12\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2 dt + 2\nu \int_0^T \alpha_x |w_{xxx}|^2 dt, \quad x \in \{0, L\}, \end{aligned} \tag{60}$$

as the boundary term, and finally,

$$\begin{aligned} R(w) := & \, \frac{\nu^4}{2} \iint_Q \left(\alpha_x^4\right)_t |w|^2 dxdt - 6\nu^2 \iint_Q \alpha_x^2 w_t w_{xx} \, dxdt - \iint_Q w_t w_{xxxx} \, dxdt \\ & - 2\nu^3 \iint_Q \left(\alpha_x^3\right)_{xxx} |w_x|^2 dxdt - 2\nu^5 \iint_Q \left(\alpha_x^5\right)_{xxx} |w|^2 dxdt. \end{aligned} \tag{61}$$

as the rest term.

Step 3. We proceed to handle the main terms defined in (56)-(59). From the weight functions defined in (44) it follows that

$$\alpha_x(t,x) = -\mu\psi'(x)\beta(t,x), \quad \alpha_{xx}(t,x) = -\mu\psi''(x)\beta(t,x) - \mu^2\psi'(x)^2\beta(t,x), \quad \forall (t,x) \in Q. \tag{62}$$

Therefore, by plugging them into the main terms defined in (56)-(59) and then considering Lemma 3.7, in particular that

$$|\psi'(x)| > 0 \quad \forall x \in \overline{(0,L)} \setminus \omega_0,$$

we see that there exist $C > 0$ such that

$$\begin{aligned} \sum_{k=0}^{3} M_k(w) \geq & \, C\|w\|_A - \frac{1}{\mu}\|w\|_A \\ & - \iint_{Q_{\omega_0}} \left(\nu^7\mu^8\beta^7|w|^2 + \nu^5\mu^6\beta^5|w_x|^2 + \nu^3\mu^4\beta^3|w_{xx}|^2 + \nu\mu^2\beta|w_{xxx}|^2\right) dxdt. \end{aligned} \tag{63}$$

Let us handle the last three terms of the right-hand side of this inequality. To this end, let us consider a non-negative function $\chi \in C_0^\infty(\omega_1)$ such that $\chi(x) = 1$ for every $x \in \omega_0$. Recall that $\overline{\omega_0} \subset \omega_1 \subseteq \omega$. Some integration by parts gives us

$$\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi|w_x|^2 dxdt = \underbrace{\frac{1}{2}\iint_{Q_{\omega_1}} \nu^5\mu^6\left(\beta^5\chi\right)_{xx}|w|^2 dxdt}_{A_1} - \underbrace{\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi w_{xx} w\, dxdt}_{A_2}. \quad (64)$$

On the one hand, the property $\alpha_x(t,x) = -\beta_x(t,x)$ for every $(t,x) \in Q$ together with (62) and Lemma 3.12 (b) allow us to obtain

$$\begin{aligned} A_1 &= \frac{1}{2}\iint_{Q_{\omega_1}} \nu^5\mu^6\left(20\beta^3\beta_x^2\chi + 5\beta^4\beta_{xx}\chi + 10\beta^4\beta_x\chi' + \beta^5\chi''\right) dxdt,\\ &\le \frac{C}{\nu^2}\left(1 + \frac{1}{\mu} + \frac{1}{\mu^2}\right)\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^5|w| dxdt. \end{aligned}$$

Therefore, if in this inequality we take into account Lemma 3.12 (a), then we have that

$$A_1 \le \frac{C}{\mu^2}\iint_Q \nu^7\mu^8\beta^7|w|^2 dxdt, \quad \forall \mu \ge 1, \quad \forall \nu \ge T^2.$$

On the other hand, the Cauchy-Schwarz inequality leads us to

$$A_2 \le \frac{C}{\varepsilon_1}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2 dxdt + C\varepsilon_1\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2 dxdt, \quad \forall \varepsilon_1 > 0.$$

Accordingly, from the combination of the two above inequalities with (64) we get

$$\begin{aligned} \iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi|w_x|^2 dxdt \le \frac{C}{\mu^2}\iint_Q \nu^7\mu^8\beta^7|w|^2 dxdt + \frac{C}{\varepsilon_1}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2 dxdt\\ + C\varepsilon_1\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2 dxdt, \quad \forall \varepsilon_1 > 0, \quad \forall \mu \ge 1, \quad \forall \nu \ge T^2. \end{aligned}$$

Note that the same arguments presented above can be applied to

$$\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2 dxdt, \quad \iint_{Q_{\omega_1}} \nu\mu^2\beta\chi|w_{xxx}|^2 dxdt,$$

to obtain the following inequalities.

$$\begin{aligned} \iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2 dxdt \le \frac{C}{\mu^2}\iint_Q \nu^5\mu^6\beta^5|w_x|^2 dxdt + \frac{C}{\varepsilon_2}\iint_Q \nu\mu^2\beta|w_{xxx}|^2 dxdt\\ + C\varepsilon_2\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi|w_x|^2 dxdt, \quad \forall \varepsilon_2 > 0, \quad \forall \mu \ge 1, \quad \forall \nu \ge T^2, \end{aligned}$$

$$\begin{aligned} \iint_{Q_{\omega_1}} \nu\mu^2\beta\chi|w_{xxx}|^2 dxdt \le \frac{C}{\mu^2}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2 dxdt + \frac{C}{\varepsilon_3}\iint_Q \frac{|w_{xxxx}|^2}{\nu\beta} dxdt\\ + C\varepsilon_3\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2 dxdt, \quad \forall \varepsilon_3 > 0, \quad \forall \mu \ge 1, \quad \forall \nu \ge T^2. \end{aligned}$$

The positive parameters ε_1, ε_2 and ε_3 can be choosen in such a way that the suitable combination of the three above inequalities yield

$$\iint_{Q_{\omega_1}} \left(\nu^5\mu^6\beta^5\chi|w_x|^2 + \nu^3\mu^4\beta^3\chi|w_{xx}|^2 + \nu\mu^2\beta\chi|w_{xxx}|^2\right) dxdt \leq \frac{C}{\mu^2}\left(1+\varepsilon+\varepsilon^3\right)\|w\|_A$$
$$+\frac{C}{\varepsilon}\left(\|w\|_A + \|w\|_B\right) + C(\varepsilon+\varepsilon^3+\varepsilon^7)\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2 dxdt, \quad \forall \varepsilon>0, \quad \forall \mu\geq 1, \quad \forall \nu \geq T^2.$$

Therefore, from the combination of this inequality with

$$\iint_{Q_{\omega_0}} \nu^{7-2n}\mu^{8-2n}\beta^{7-2n}\left|\frac{\partial^n w}{\partial x^n}\right|^2 dxdt \leq \iint_{Q_{\omega_1}} \nu^{7-2n}\mu^{8-2n}\beta^{7-2n}\chi\left|\frac{\partial^n w}{\partial x^n}\right|^2 dxdt, \quad n=1,2,3,$$

and then the resulting expression with (63), we see that by setting $\varepsilon=\varepsilon_0$ with $\varepsilon_0\geq 1$ large enough give us

$$\sum_{k=0}^{3} M_k(w) \geq C\|w\|_A - \frac{1}{\varepsilon_0}\|w\|_B - \varepsilon_0^7\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2 dxdt, \quad \forall \mu\geq \varepsilon_0^3, \quad \forall \nu\geq T^2. \tag{65}$$

Step 4. We proceed to handle the boundary term defined in (60). The Cauchy-Schwarz inequality and Lemma 3.12 allow us to obtain the following inequalities that are valid at $x\in\{0,L\}$.

- $$\left|2\nu^3\int_0^T \left(\alpha_x^3\right)_{xx}|w_x|^2dt\right| = \left|2\nu^3\int_0^T \left(6\alpha_x\alpha_{xx}^2 + 3\alpha_x^2\alpha_{xxx}\right)|w_x|^2dt\right|,$$
$$\leq \frac{C}{\nu^2}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)^2\frac{1}{\mu^2}\int_0^T \nu^5\mu^5\beta^5|w_x|^2dt, \quad \forall\mu\geq 1.$$

- $$\left|4\nu^3\int_0^T \left(\alpha_x^3\right)_x w_xw_{xx}\,dt\right| = \left|12\nu^3\int_0^T \alpha_x^2\alpha_{xx}w_xw_{xx}\,dt\right|,$$
$$\leq C\nu^4\int_0^T \alpha_x^2\alpha_{xx}^2|w_x|^2dt + C\nu^2\int_0^T\alpha_x^2|w_{xx}|^2dt,$$
$$\leq \frac{C}{\nu}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)\frac{1}{\mu}\int_0^T\left(\nu^5\mu^5\beta^5|w_x|^2 + \nu^3\mu^3\beta^3|w_{xx}|^2\right)dt, \quad \forall\mu\geq 1.$$

- $$\left|4\nu^3\int_0^T \alpha_x^3 w_xw_{xxx}\,dt\right| \leq 4\nu^5\int_0^T|\alpha_x^5||w_x|^2dt + \nu\int_0^T|\alpha_x||w_{xxx}|^2dt.$$

- $$\left|2\nu^5\int_0^T \left(\alpha_x^5\right)_{xx}|w|^2dt\right| = \left|2\nu^5\int_0^T\left(20\alpha_x^3\alpha_{xx}^2 + 5\alpha_x^4\alpha_{xxx}\right)|w|^2dt\right|,$$
$$\leq \frac{C}{\nu^2}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)^2\frac{1}{\mu^2}\int_0^T\nu^7\mu^7\beta^7|w|^2dt, \quad \forall\mu\geq 1.$$

- $$\left|4\nu^5\int_0^T\left(\alpha_x^5\right)_x ww_x\,dt\right| = \left|20\nu^5\int_0^T\alpha_x^4\alpha_{xx}ww_x\,dt\right|,$$
$$\leq C\nu^6\int_0^T\alpha_x^4\alpha_{xx}^2|w|^2dt + C\nu^4\int_0^T\alpha_x^4|w_x|^2dt,$$
$$\leq \frac{C}{\nu}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)\frac{1}{\mu}\int_0^T\left(\nu^7\mu^7\beta^7|w|^2 + \nu^5\mu^5\beta^5|w_x|^2\right)dt, \quad \forall\mu\geq 1.$$

- $$\left|4\nu^5\int_0^T\alpha_x^5ww_{xx}\,dt\right| \leq \nu^7\int_0^T|\alpha_x^7||w|^2dt + 4\nu^3\int_0^T|\alpha_x^3||w_{xx}|^2dt.$$

An useful feature of the function $\psi \in C^4([0,L])$ given by Lemma 3.7 is that $\psi'(0) > 0$ and $\psi'(L) < 0$. In fact, this is a consequence of

$$\psi(x) > 0 \ \ \forall x \in (0,L), \quad \psi(0) = \psi(L) = 0.$$

From this feature and the combination of the six above inequalities with the boundary term defined in (60) it follows that the choice of $\mu_0 \geq 1$ large enough gives us

$$B(w,0) \leq 0, \quad B(w,1) \geq 0, \quad \forall \mu \geq \mu_0, \quad \forall \nu \geq T^2. \tag{66}$$

Step 5. Before handling the rest term defined in (61), we are going to apply Theorem 3.8 to the equation satisfied by $z := q_{xx} + \lambda q$. Let $\omega_{1/2}$ be a domain such that $\overline{\omega_0} \subset \omega_{1/2}$ and $\overline{\omega_{1/2}} \subset \omega_1$. Since z satisfies the adjoint equation (18) with a final state in $H_0^2(0,L)$, the above theorem in particular yields

$$\begin{aligned}
&\iint_Q e^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|z|^2 + \nu^5\mu^6\beta^5|z_x|^2 + \nu^3\mu^4\beta^3|z_{xx}|^2\right)dxdt \\
&\quad\leq C\iint_Q e^{-2\nu\alpha}|Lz|^2dxdt + C\iint_{Q_{\omega_{1/2}}} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|z|^2dxdt, \quad \forall \mu \geq \mu_0, \quad \forall \nu \geq \max\{T,T^2\}.
\end{aligned}$$

Let us consider a non-negative function $\chi \in C_0^\infty(\omega_1)$ such that $\chi(x) = 1$ for every $x \in \omega_{1/2}$. Then, by using the same arguments as those used in Step 3 we obtain

$$\begin{aligned}
\iint_{Q_{\omega_{1/2}}} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q_{xx}|^2dxdt &\leq \frac{C}{\varepsilon}\iint_Q e^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q_{xx}|^2 + \nu^5\mu^6\beta^5|q_{xxx}|^2\right)dxdt \\
&\quad + C(1+\varepsilon)(1+\varepsilon^2)\iint_{Q_{\omega_1}} e^{-2\nu\alpha}\nu^{11}\mu^{12}\beta^{11}|q|^2dxdt, \quad \forall \varepsilon > 0, \quad \forall \mu \geq 1, \quad \forall \nu \geq T^2.
\end{aligned}$$

Therefore, by combining $z := q_{xx} + \lambda q$ and $Lz = (Lq)_{xx} + \lambda Lq$ with the two above inequalities we see that $\varepsilon > 0$ can be choosen in such a way that for $\mu_0 \geq 1$ large enough we have

$$\begin{aligned}
&\iint_Q e^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q_{xx}|^2 + \nu^5\mu^6\beta^5|q_{xxx}|^2 + \nu^3\mu^4\beta^3|q_{xxxx}|^2\right)dxdt \leq \\
&+C\iint_Q e^{-2\nu\alpha}\left(|(Lq)_{xx}|^2 + |Lq|^2\right)dxdt + C\iint_Q e^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2\right)dxdt \\
&\qquad + C\iint_{Q_{\omega_1}} e^{-2\nu\alpha}\nu^{11}\mu^{12}\beta^{11}|q|^2dxdt, \quad \forall \mu \geq \mu_0, \quad \forall \nu \geq \max\{T,T^2\}.
\end{aligned}$$

Moreover, in virtue of $Lq = -q_t + q_{xxxx} + \lambda q_{xx}$ and Lemma 3.7 (a) we actually have

$$\begin{aligned}
&\iint_Q e^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q_{xx}|^2 + \nu^5\mu^6\beta^5|q_{xxx}|^2 + \nu^3\mu^4\beta^3|q_{xxxx}|^2 + \nu^3\mu^4\beta^3|q_t|^2\right)dxdt \leq \\
&+C\iint_Q e^{-2\nu\alpha}\left(|(Lq)_{xx}|^2 + \nu^3\mu^4\beta^3|Lq|^2\right)dxdt + C\iint_Q e^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2 + \nu^5\mu^6\beta^5|q_x|^2\right)dxdt \\
&\qquad + C\iint_{Q_{\omega_1}} e^{-2\nu\alpha}\nu^{11}\mu^{12}\beta^{11}|q|^2dxdt, \quad \forall \mu \geq \mu_0, \quad \forall \nu \geq \max\{T,T^2\}.
\end{aligned} \tag{67}$$

Step 6. We proceed to handle the rest term defined in (61) and the term $\|P_3 w\|^2_{L^2(Q)}$. All of these terms can be handled as we did in Step 4, nevertheless, we pay special attention to the first three terms of the right-hand side of (61) and to the term $\nu\alpha_t w$ in $P_3 w$.

The Cauchy-Schwarz inequality, (62) and Lemma 3.12 (a) allow us to obtain the following inequalities.

- $$\frac{\nu^4}{2}\iint_Q \left(\alpha_x^4\right)_t |w|^2 dxdt = 2\nu^4 \iint_Q \alpha_x^3 \alpha_{xt} |w|^2 dxdt,$$
$$\leq \frac{C}{\nu^3}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)^2 T\frac{1}{\mu^7}\iint_Q \nu^7\mu^8\beta^7|w|^2dxdt. \tag{68}$$

Here we have used the fact that $|\alpha_{xt}(t,x)| \leq CT\beta^2(t,x)$ hold for every $(t,x)\in Q$.

- $$6\nu^2\iint_Q \alpha_x^2 w_t w_{xx}\, dxdt \leq C\iint_Q \nu^2\mu^2\beta^2|w_t|^2dxdt + \frac{C}{\nu}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)\frac{1}{\mu^3}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2dxdt.$$

- $$\iint_Q w_t w_{xxxx}\, dxdt \leq C\iint_Q \nu^2\mu^2\beta^2|w_t|^2dxdt + C\iint_Q \frac{|w_{xxxx}|^2}{\nu^2\mu^2\beta^2}dxdt.$$

- $$\nu^2\iint_Q \alpha_t^2|w|^2dxdt \leq \frac{C}{\nu^5}\left(\frac{T^2}{8\|\psi\|_{L^\infty(0,1)}}\right)^3 T^2\frac{1}{\mu^{10}}\iint_Q \nu^7\mu^8\beta^7|w|^2dxdt. \tag{69}$$

Here we have used the fact that $|\alpha_t(t,x)| \leq T\beta^2(t,x)$ hold for every $(t,x)\in Q$. Accordingly, from the combination of the above inequalities with the rest term defined in (61) and $P_3 w$ we get

$$|R(w)| \leq \frac{C}{\mu}\|w\|_A + C\iint_Q \nu^2\mu^2\beta^2|w_t|^2dxdt + C\iint_Q \frac{|w_{xxxx}|^2}{\nu^2\mu^2\beta^2}dxdt, \quad \forall\mu\geq 1, \quad \forall\nu\geq\max\{T,T^2\}. \tag{70}$$

$$\|P_3 w\|^2_{L^2(Q)} \leq \frac{C}{\mu}\|w\|_A, \quad \forall\mu\geq 1, \quad \forall\nu\geq\max\{T,T^2\}. \tag{71}$$

The above relation between ν and $\max\{T,T^2\}$ was asked because of (68) and (69).

Step 7. We proceed to obtain the desired Carleman estimate. First, from the combination of (66) and (65) with (55), and then the resulting expression with (53), it follow that for $\varepsilon_0\geq 1$ large enough we have

$$C\left(\|P_1 w\|_{L^2(Q)} + \|P_2 w\|_{L^2(Q)} + \|w\|_A\right) \leq \|Pw - P_3 w\|^2_{L^2(Q)} + \frac{1}{\varepsilon_0}\|w\|_B + |R(w)|$$
$$+\varepsilon_0^7\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2dxdt, \quad \forall\mu\geq\varepsilon_0^3, \quad \forall\nu\geq T^2.$$

Hence, from this inequality and (54) we see that by taking $\varepsilon_0\geq 1$ large enough gives us the existence of a $\mu_0\geq 1$ also large enough such that

$$C\left(\|w\|_A + \|w\|_B\right) \leq \|Pw - P_3 w\|^2_{L^2(Q)} + |R(w)| + \iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2dxdt, \quad \forall\mu\geq\mu_0, \quad \forall\nu\geq T^2.$$

Second, plugging (70) and (71) into this inequality and then taking $\mu_0\geq 1$ large enough lead us to

$$C\left(\|w\|_A+\|w\|_B\right)\le \|Pw\|^2_{L^2(Q)}+\iint_{Q_{\omega_1}}\nu^7\mu^8\beta^7|w|^2dxdt$$
$$+\iint_Q \nu^2\mu^2\beta^2|w_t|^2dxdt+\iint_Q\frac{|w_{xxxx}|^2}{\nu^2\mu^2\beta^2}dxdt,\quad \forall\mu\ge\mu_0,\quad \forall\nu\ge\max\{T,T^2\}. \tag{72}$$

Third, since the inequalities

- $\displaystyle\iint_Q e^{-2\nu\alpha}\frac{1}{\nu\beta}\left(|(e^{\nu\alpha}w)_t|^2+|(e^{\nu\alpha}w)_{xxxx}|^2\right)dxdt\le C\left(\|w\|_A+\|w\|_B\right).$
- $\displaystyle\sum_{n=0}^{3}\iint_Q e^{-2\nu\alpha}\nu^{7-2n}\mu^{8-2n}\beta^{7-2n}\left|\frac{\partial^n(e^{\nu\alpha}w)}{\partial x^n}\right|^2dxdt\le C\|w\|_A.$
- $|w_t|^2\le Ce^{-2\nu\alpha}\left(|q_t|^2+T^2\nu^2\beta^4|q|^2\right),\quad \forall(t,x)\in Q.$
- $|w_{xxxx}|^2\le Ce^{-2\nu\alpha}\left(\nu^8\mu^8\beta^8|q|^2+\nu^6\mu^6\beta^6|q_x|^2+\nu^4\mu^4\beta^4|q_{xx}|^2+\nu^2\mu^2\beta^2|q_{xxx}|^2\right),\quad \forall(t,x)\in Q.$

hold for every $\mu\ge\mu_0$, with $\mu_0\ge 1$ large enough, and $\nu\ge\max\{T,T^2\}$, we see that from (72), $w=e^{-\nu\alpha}q$ and $Pw=e^{-\nu\alpha}L(e^{\nu\alpha}w)$ it follow

$$\begin{aligned}&\iint_Q e^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2+\nu^5\mu^6\beta^5|q_x|^2+\nu^3\mu^4\beta^3|q_{xx}|^2+\nu\mu^2\beta|q_{xxx}|^2\right)dxdt\\
&+\iint_Q e^{-2\nu\alpha}\left(\frac{|q_t|^2+|q_{xxxx}|^2}{\nu\beta}\right)dxdt\le C\iint_Q e^{-2\nu\alpha}|Lq|^2dxdt+C\iint_{Q_{\omega_1}}e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt,\\
&+\frac{C}{\mu^2}\iint_Q e^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2+\nu^5\mu^6\beta^5|q_x|^2+\nu^3\mu^4\beta^3|q_{xx}|^2+\nu\mu^2\beta|q_{xxx}|^2\right)dxdt\\
&\qquad+\frac{C}{\mu^2}\iint_Q e^{-2\nu\alpha}\nu^3\mu^4\beta^3|q_t|^2dxdt,\quad\forall\mu\ge\mu_0,\quad\forall\nu\ge\max\{T,T^2\}.\end{aligned}$$

Finally, from the combination of this inequality with (67) we obtain (46) by choosing $\mu_0\ge 1$ large enough. The proof of Proposition 3.9 is complete. ■

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