Análisis espectral, Teoremas de tipo Ingham y aplicaciones a problemas de control[1]

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Seminario Control en Tiempos de Crisis

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[1] Joint works with Mercado, Queiroz-Souza, Souza, de Teresa

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Nonlinear Mindlin-Timoshenko system

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k(\phi + \psi_x) = 0 & \text{in} \quad Q, \\ \rho h \psi_{tt} - k(\phi + \psi_x)_x - \left[\psi_x\left(\eta_x + \dfrac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{in} \quad Q, \\ \rho h \eta_{tt} - \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)_x = 0 & \text{in} \quad Q, \end{cases} \qquad \text{(M-T)}$$

- $Q = (0, L) \times (0, T)$
- ϕ - angle of rotation
- ψ - vertical displacement
- η - longitudinal displacement
- ρ - density, h - thickness of the beam
- $k > 0$ - modulus of elasticity in shear

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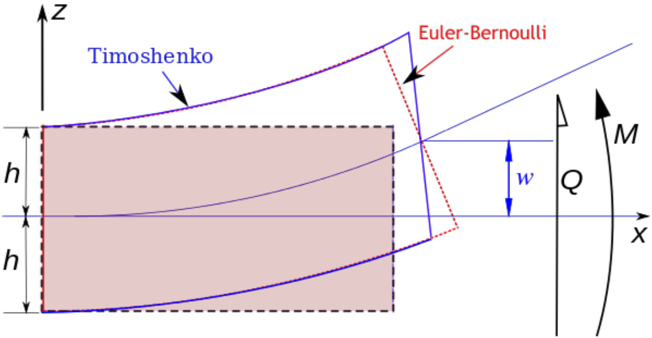
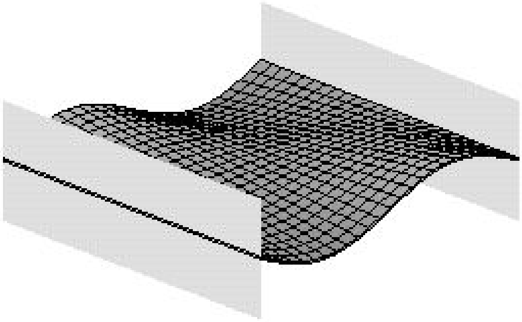
[2] Mindlin (1951) , Timoshenko-Woinowsky (1959) , Lagnese-Lions (1988)

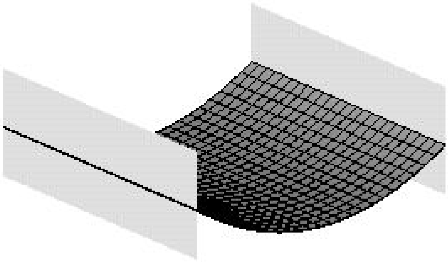
Timoshenko vs Euler-Bernoulli





(video)

When assuming that the linear filament of the beam remains perpendicular to the deformed middle surface, the transverse shear effects are neglected and ...



von Kármán system

... we obtain the so called von Kármán system:

$$\begin{cases} \rho h \psi_{tt} - \dfrac{\rho h^3}{12}\psi_{xxtt} + \psi_{xxxx} - \left[\psi_x\left(\eta_x + \dfrac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{in} \quad Q, \\ \rho h \eta_{tt} - \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)_x = 0 & \text{in} \quad Q. \end{cases} \tag{vK}$$

Neglecting the shear effects of the beam is equivalent to making $k \to \infty$ in $(M - T)$.
[3]

[3] Lagnese-Lions (1988)

Lagnese-Lions conjecture

Conjecture: *The Mindlin-Timoshenko system* $(M - T)$ *approaches, in some sense, the von Kármán system as* $k \to \infty$.

The conjecture is true: FA, Braz e Silva, Queiroz-Souza (Anal. PDE'18)

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Linear Mindlin-Timoshenko system

$$
\begin{cases}
\frac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k(\phi + \psi_x) = 0 & \text{in} \quad Q, \\
\rho h \psi_{tt} - k(\phi + \psi_x)_x = 0 & \text{in} \quad Q, \\
\phi(0,\cdot) = \phi(L,\cdot) = \psi_x(0,\cdot) = \psi_x(L,\cdot) = 0 & \text{in} \quad (0,T) \\
\phi(\cdot,0) = \phi_0, \quad \phi_t(\cdot,0) = \phi_1 & \text{in} \quad (0,L), \\
\psi(\cdot,0) = \psi_0, \quad \psi_t(\cdot,0) = \psi_1 & \text{in} \quad (0,L).
\end{cases}
\qquad \text{(M-T)}
$$

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$$\Big\downarrow \; k \to \infty$$

[4] Mindlin (1951), Timoshenko-Woinowsky (1959), Lagnese-Lions (1988)

Linear Kirchhoff system

$$\begin{cases} \rho h \psi_{tt} - \dfrac{\rho h^3}{12}\psi_{xxtt} + \psi_{xxxx} = 0 & \text{in} \quad Q, \\ \psi_x(0,\cdot) = \psi_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\ \psi_{xxx}(0,\cdot) = \psi_{xxx}(L,\cdot) = 0 & \text{in} \quad (0,T), \\ \psi(\cdot,0) = \psi_0, \quad \psi_t(\cdot,0) = \psi_1 & \text{in} \quad (0,L). \end{cases} \tag{K}$$

[5]

[5] Lagnese-Lions (1988)

The energy of the Mindlin-Timoshenko system

$$E_k(t) = \frac{1}{2}\int_0^L \left\{ \frac{\rho h^3}{12} |\phi_t(x,t)|^2 + \rho h |\psi_t(x,t)|^2 + |\phi_x(x,t)|^2 + \right.$$
$$\left. + k|\phi(x,t) + \psi_x(x,t)|^2 \right\} dx$$

is conservative, that is,

$$E_k(t) = E_k(0).$$

Control problem

$$\begin{cases} \dfrac{\rho h^3}{12} u_{tt} - u_{xx} + k\left(u + v_x\right) = 0 & \text{in} \quad Q, \\ \rho h v_{tt} - k\left(u + v_x\right)_x + f(v) = 0 & \text{in} \quad Q, \\ u\left(0,\cdot\right) = 0, \quad u\left(L,\cdot\right) = 0 & \text{in} \quad \left(0,T\right), \\ v_x\left(0,\cdot\right) = \Theta_k, \quad v_x\left(L,\cdot\right) = 0 & \text{in} \quad \left(0,T\right), \\ u\left(\cdot,0\right) = u_0, \quad u_t\left(\cdot,0\right) = u_1 & \text{in} \quad \left(0,L\right), \\ v\left(\cdot,0\right) = v_0, \quad v_t\left(\cdot,0\right) = v_1 & \text{in} \quad \left(0,L\right), \end{cases}$$

where

$$\lim_{|s|\to+\infty} \frac{f(s)}{s} = \alpha, \quad \alpha \geq -\left(\frac{\pi}{L}\right)^4.$$

Control problem

Problem: given $T > 0$, large enough, initial data $\{u_0, u_1, v_0, v_1\}$ and final data $\{\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1\}$, to find a control Θ_k such that the solution of system satisfies the conditions

$$u(\cdot, T) = \hat{u}_0, \;\; u_t(\cdot, T) = \hat{u}_1, \;\; v(\cdot, T) = \hat{v}_0, \;\; v_t(\cdot, T) = \hat{v}_1 \quad \text{in} \quad (0, L).$$

Control problem

$$\begin{cases} \rho h v_{tt} - \dfrac{\rho h^3}{12} v_{xxtt} + v_{xxxx} + f(v) = 0 & \text{in} \quad Q, \\ v_x(0,\cdot) = 0, \quad v_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\ v_{xxx}(0,\cdot) = \Theta, \quad v_{xxx}(L,\cdot) = 0 & \text{in} \quad (0,T), \\ v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1 & \text{in} \quad (0,L). \end{cases}$$

Control problem

Problem: given $T > 0$, large enough, initial data $\{v_0, v_1\}$ and final data $\{\hat{v}_0, \hat{v}_1\}$, to find a control Θ such that the solution of system satisfies the conditions

$$v(\cdot, T) = \hat{v}_0, \;\; v_t(\cdot, T) = \hat{v}_1 \quad \text{in} \quad (0, L).$$

Motivation – Lagnese-Lions book (1988)

The goals for the linear system are:

(i) to show that the control time T is independent of k, for any given initial state, and to find, for each k, a control Θ_k driving the M-T system to rest at time T, and

(ii) to study the behavior of Θ_k as $k \to \infty$.

Conjecture: *As $k \to \infty$, Θ_k converges, in some appropriate sense, towards a control driving the Kirchhoff system to equilibrium in time T.*

Results for the linear system [6]:

- The controls Θ_k of the M-T system may diverge exponentially as $k \to \infty$.
- The exact controllability requirement on M-T system is relaxed to a partial controllability property over a suitable projection of solutions, and the controls Θ_k remain bounded as $k \to \infty$.
- The partial controls Θ_k obtained this way converge to an exact control for the limit Kirchhoff system.

[6] A-Zuazua (SIAM J. Control Optim.'08)

Strategy

Linearization

$+$

Fixed point argument

Linearized system

$$\begin{cases} \dfrac{\rho h^3}{12} u_{tt} - u_{xx} + k\left(u + v_x\right) = 0 & \text{in} \quad Q, \\ \rho h v_{tt} - k\left(u + v_x\right)_x + \alpha v = 0 & \text{in} \quad Q, \\ u\left(0, \cdot\right) = 0, \quad u\left(L, \cdot\right) = 0 & \text{in} \quad \left(0, T\right), \\ v_x\left(0, \cdot\right) = \Theta_k, \quad v_x\left(L, \cdot\right) = 0 & \text{in} \quad \left(0, T\right), \\ u\left(\cdot, 0\right) = u_0, \quad u_t\left(\cdot, 0\right) = u_1 & \text{in} \quad \left(0, L\right), \\ v\left(\cdot, 0\right) = v_0, \quad v_t\left(\cdot, 0\right) = v_1 & \text{in} \quad \left(0, L\right), \end{cases}$$

HUM - J.-L. Lions

Hilbert Uniqueness Method (HUM)

$$\Downarrow$$

Controllability $\Longleftrightarrow$ Observability to a adjoint system

Adjoint system

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k\left(\phi + \psi_x\right) = 0 & \text{in} \quad Q, \\ \rho h \psi_{tt} - k\left(\phi + \psi_x\right)_x + \alpha\psi = 0 & \text{in} \quad Q, \\ \phi\left(0,\cdot\right) = \phi\left(L,\cdot\right) = \psi_x\left(0,\cdot\right) = \psi_x\left(L,\cdot\right) = 0 & \text{in} \quad \left(0,T\right), \\ \phi\left(\cdot,0\right) = \phi_0, \quad \phi_t\left(\cdot,0\right) = \phi_1 & \text{in} \quad \left(0,L\right), \\ \psi\left(\cdot,0\right) = \psi_0, \quad \psi_t\left(\cdot,0\right) = \psi_1 & \text{in} \quad \left(0,L\right). \end{cases}$$

Nonuniform observability

Theorem (A, Souza, Queiroz-Souza (to appear))

For $T > 2\beta L$, with

$$\beta = \max\left\{\sqrt{\frac{\rho h^3}{12}}, \sqrt{\frac{\rho h}{k}}\right\}, \ k \geq 1, \ \text{and } h \leq \min\left\{\sqrt[3]{\frac{3}{\rho}}, \frac{1}{4\rho}\right\},$$

there exists a constant $C_k^* > 0$ such that

$$E_k(0) \leq C_k \int_0^T \left\{ |\phi_x(0,t)|^2 + |\psi(0,t)|^2 + \rho h |\psi_t(0,t)|^2 \, dt \right\},$$

where $C_k \sim o(e^{\sqrt{k}})$.

Spectral analysis

$$\Phi_t = -i\mathcal{A}\Phi,$$

$\Phi = [\phi, \phi_t, \psi, \psi_t]^T$, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$

$$\mathcal{A} = i \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{12}{\rho h^3}\left(\frac{\partial^2}{\partial x^2} - k\right) & 0 & -\frac{12k}{\rho h^3}\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{\rho h}\frac{\partial}{\partial x} & 0 & \frac{1}{\rho h}(k\frac{\partial^2}{\partial x^2} - \alpha) & 0 \end{bmatrix}$$

with domain

$$D(\mathcal{A}) = \left[H_0^1(0,L) \cap H^2(0,L)\right] \times H_0^1(0,L) \times \mathcal{H}_N^2 \times H^1(0,L)$$

and

$$\mathcal{X} = H_0^1(0,L) \times L^2(0,L) \times H^1(0,L) \times L^2(0,L).$$

$$\mathcal{A}U = \lambda U.$$

In view of the various equations involved and the boundary conditions satisfied by the components ϕ and ψ, the solutions $U = [\phi, \phi_t, \psi, \psi_t]^T$ associated with the eigenfunctions are such that

$$\{\phi(x,t), \psi(x,t)\} = e^{-i\lambda t}\{\sin(m\pi x/L), \nu\cos(m\pi x/L)\},$$

where the constant ν is to be determined in terms of m and λ.

$$\begin{cases} \lambda^2 \dfrac{\rho h^3}{12}\phi + \phi_{xx} - k\left(\phi + \psi_x\right) = 0, \\ \lambda^2 \rho h \psi + k\left(\phi + \psi_x\right)_x - \alpha\psi = 0. \end{cases}$$

After some calculations, we get

$$\phi_{xxxx} + \left(\lambda^2 \frac{\rho h^3}{12} + \frac{\lambda^2 \rho h - \alpha}{k}\right)\phi_{xx} + \left(\frac{\lambda^2 \rho h - \alpha}{k}\right)\left(\lambda^2 \frac{\rho h^3}{12} - k\right)\phi = 0.$$

Since $\phi(x,t) = e^{-i\lambda t}\sin(m\pi x/L)$, we obtain

$$\lambda^4 - a_2\lambda^2 + a_0 = 0,$$

with

$$a_2 = \frac{12}{\rho h^3}\frac{m^2\pi^2}{L^2} + \frac{k}{\rho h}\frac{m^2\pi^2}{L^2} + \frac{12k}{\rho h^3} + \frac{\alpha}{\rho h}$$

and

$$a_0 = \frac{12k}{\rho^2 h^4}\frac{m^4\pi^4}{L^4} + \frac{12\alpha}{\rho^2 h^4}\frac{m^2\pi^2}{L^2} + \frac{12\alpha k}{\rho^2 h^4}.$$

We find

$$\widetilde{\lambda}_{k,m}^{\pm} = \pm\left[\frac{6}{\rho h^3}\frac{m^2\pi^2}{L^2} + \frac{k}{2\rho h}\frac{m^2\pi^2}{L^2} + \frac{6k}{\rho h^3} + \frac{\alpha}{2\rho h} + \frac{1}{2}\sqrt{r}\right]^{1/2}$$

and

$$\lambda_{k,m}^{\pm} = \pm\left[\frac{6}{\rho h^3}\frac{m^2\pi^2}{L^2} + \frac{k}{2\rho h}\frac{m^2\pi^2}{L^2} + \frac{6k}{\rho h^3} + \frac{\alpha}{2\rho h} - \frac{1}{2}\sqrt{r}\right]^{1/2},$$

with

$$r = \frac{144k^2}{\rho^2 h^6} + \frac{288k}{\rho^2 h^6}\frac{m^2\pi^2}{L^2} + \frac{24k^2}{\rho^2 h^4}\frac{m^2\pi^2}{L^2} + \left(\frac{12}{\rho h^3}\frac{m^2\pi^2}{L^2} - \frac{k}{\rho h}\frac{m^2\pi^2}{L^2}\right)^2 + \frac{\alpha^2}{\rho^2 h^2} - \frac{24\alpha k}{\rho^2 h^4} - \frac{24\alpha}{\rho^2 h^4}\frac{m^2\pi^2}{L^2} + \frac{2\alpha k}{\rho^2 h^2}\frac{m^2\pi^2}{L^2}.$$

For m fixed, we see easily that, as k tends to infinity,

$$\widetilde{\lambda}_{k,m}^{\pm} \to \pm\infty. \tag{1}$$

This corresponds to that half of the spectrum that disappears when letting k tend to infinity.

Proposition (A, Souza, Queiroz-Souza (to appear))

For fixed $m \in \mathbb{N}$, as $k \to \infty$,

$$\lambda_{k,m}^{\pm} \to \lambda_m^{\pm} = \pm\sqrt{\frac{12\pi^4 m^4}{12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2} + \frac{12\alpha L^2}{12\rho h L^2 + \pi^2 \rho h^3 m^2}}.$$

These are the eigenvalues of the limit Kirchhoff system for which the corresponding eigenfunctions are $(L/m\pi)\cos(m\pi x/L)$.

The eigenfunctions associated to the eigenvalues families $(\lambda^{\pm}_{k,m})_m$ and $(\widetilde{\lambda}^{\pm}_{k,m})_m$ are, respectively:

$$U^{\pm}_{k,m} = \left[\begin{array}{cccc} \sin\left(\frac{m\pi x}{L}\right) & -i\lambda^{\pm}_{k,m}\sin\left(\frac{m\pi x}{L}\right) & \nu_{k,m}\cos\left(\frac{m\pi x}{L}\right) & -i\lambda^{\pm}_{k,m}\nu_{k,m}\cos\left(\frac{m\pi x}{L}\right) \end{array}\right]^T$$

and

$$\widetilde{U}^{\pm}_{k,m} = \left[\begin{array}{cccc} \sin\left(\frac{m\pi x}{L}\right) & -i\widetilde{\lambda}^{\pm}_{k,m}\sin\left(\frac{m\pi x}{L}\right) & \widetilde{\nu}_{k,m}\cos\left(\frac{m\pi x}{L}\right) & -i\widetilde{\lambda}^{\pm}_{k,m}\widetilde{\nu}_{k,m}\cos\left(\frac{m\pi x}{L}\right) \end{array}\right]^T,$$

with

$$\nu_{k,m} = \frac{m\pi}{Lk} + \frac{L}{m\pi} - \frac{\rho h^3 L\lambda^2_{k,m}}{12m\pi k} \quad \text{and} \quad \widetilde{\nu}_{k,m} = \frac{m\pi}{Lk} + \frac{L}{m\pi} - \frac{\rho h^3 L\widetilde{\lambda}^2_{k,m}}{12m\pi k}.$$

Let us introduce a closed subspaces of $\mathcal{X}$

$$\mathcal{X}_k = \left\{ \Phi_0 \in \mathcal{X} : \Phi_0 = \sum_{m\in\mathbb{N}} \left(a^0_{k,m} U^+_{k,m} + b^0_{k,m} U^-_{k,m} \right) \right\},$$

$$\widetilde{\mathcal{X}}_k = \left\{ \Phi_0 \in \mathcal{X} : \Phi_0 = \sum_{m\in\mathbb{N}} \left(\widetilde{a}^0_{k,m} \widetilde{U}^+_{k,m} + \widetilde{b}^0_{k,m} \widetilde{U}^-_{k,m} \right) \right\}.$$

$$\mathcal{X} = \mathcal{X}_k \oplus \widetilde{\mathcal{X}}_k.$$

$$\mathcal{X}_\infty = \left\{ \Phi_0 \in \mathcal{X} : \Phi_{0k} \to \Phi_0 \text{ weakly in } \mathcal{X}, \text{ where } \Phi_{0k} \in \mathcal{X}_k, \ \forall k \geq 1 \right\}.$$

Proposition (A, Souza, Queiroz-Souza (to appear))

The following inclusion holds:

$$\mathcal{X}_\infty \subset \{\Phi_0 = \{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{X} : \{\phi_0, \phi_1, \psi_0, \psi_1\} = \{-\psi_{0x}, -\psi_{1x}, \psi_0, \psi_1\}\}.$$

Proposition (A, Souza, Queiroz-Souza (to appear))

Let $(\Phi_{0k})_{k\geq 1}$ *be a sequence with* $\Phi_{0k} = \{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\} \in \mathcal{X}_k$, *for all* $k \geq 1$, *such that* $\Phi_{0k} \to \Phi_0$ *weakly in* $\mathcal{X}$, *where* $\Phi_0 = \{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{X}_\infty$. *Then*

$$(k(\phi_{0k} + \psi_{0kx}))_{k\geq 1} \text{ is bounded in } L^2(0, L).$$

Uniform observability

Theorem (Haraux (JMPA'89))

Let $f = f(t)$ be of the form $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t}$, where $(\lambda_n)_n$ is a sequence of real numbers such that there exist $N \in \mathbb{N}$, $\gamma > 0$, and $\gamma_\infty > 0$ such that

$$\lambda_{n+1} - \lambda_n \geq \gamma_\infty > 0 \quad \textit{if} \quad |n| > N,$$

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0 \quad \forall n \in \mathbb{Z}.$$

Let $T > 0$ be such that $T > 2\pi/\gamma_\infty$. Then, there exist two positive constants $C^{(1)}$ and $C^{(2)}$ such that

$$C^{(1)} \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \int_0^T |f(t)|^2 \, dt \leq C^{(2)} \sum_{n\in\mathbb{Z}} |a_n|^2 .$$

Uniform gap for the family $(\lambda^{\pm}_{k,m})$

Theorem (A, Souza, Queiroz-Souza (to appear))

Given $0 < \epsilon < \epsilon_0$ *and* $k \geq k(\epsilon)$, *there exists* $m_0 = m_0(\epsilon)$ *such that*

$$\left|\lambda^{\pm}_{k,m+1} - \lambda^{\pm}_{k,m}\right| \geq \gamma_\infty > 0 \quad \textit{with} \quad \gamma_\infty = \frac{\pi}{L}\sqrt{\frac{12}{\rho h^3}} - \epsilon, \quad \forall m \geq m_0,$$

Uniform observability - Projected solutions

Theorem (A, Souza, Queiroz-Souza (to appear))

Let $T > 2L\sqrt{\rho h^3/12}$. Then there exist positive constants $c = c(T)$ and $C = C(T)$ such that

$$cE_k(0) \leq \int_0^T |\psi_t(0,t)|^2 \, dt \leq CE_k(0).$$

$$\psi(x,t) = \sum_{m\in\mathbb{N}} \nu_{km} A_m e^{i\lambda_{km}t} \cos\left(\frac{m\pi x}{L}\right)$$

Gap for the family $(\widetilde{\lambda}^{\pm}_{k,m})$

Theorem (A, Souza, Queiroz-Souza (to appear))

Given $0 < \epsilon < \widetilde{\epsilon}_0$ *and* $k \geq \widetilde{k}(\epsilon)$, *there exists* $\widetilde{m}_0 = \widetilde{m}_0(k, \epsilon)$ *such that*

$$\left|\widetilde{\lambda}^{\pm}_{k,m+1} - \widetilde{\lambda}^{\pm}_{k,m}\right| \geq \widetilde{\gamma}_\infty > 0 \quad \textit{with} \quad \widetilde{\gamma}_\infty = \frac{\pi\sqrt{k}}{L\sqrt{\rho h}} - \epsilon, \quad \forall m \geq \widetilde{m}_0,$$

NEW nonuniform observability - All solutions

Theorem (A, Souza, Queiroz-Souza (to appear))

Let $T > 2\beta L$, with $\beta = \max\{\sqrt{\rho h^3/12}, \sqrt{\rho h/k}\}$. Then there exist positive constants $c = c(k, T)$ and $C = C(k, T)$ such that

$$cE_k(0) \leq \int_0^T |\psi_t(0,t)|^2 \, dt \leq CE_k(0).$$

$$\psi(x,t) = \sum_{m \in \mathbb{N}} \left(\nu_{km} A_m e^{i\lambda_{km} t} + \widetilde{\nu}_{km} \widetilde{A}_m e^{i\widetilde{\lambda}_{km} t} \right) \cos\left(\frac{m\pi x}{L}\right)$$

Comments and open problem

- To analyze whether the same control results hold when the nonlinearity f has behavior at infinity like $-s \log^p |s|$.
  - For $p \in [1, 3/2]$ the (M–T) system is exactly controllable by considering TWO controls[7].
  - For $p > 2$ the (M–T) system is not exactly controllable.
  - For $p \in (3/2, 2]$ we have an interesting and open problem.
  - To analyze whether the exact controllability property of the nonlinear Kirchhoff equation can be obtained as a limit of the (M–T) system.

[7] A-Antunes-Mercado (JMAA'19)

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Thermoelastic system

$$
\begin{cases}
\dfrac{\rho h^3}{12} u_{tt} - u_{xx} + k\left(u + v_x\right) + \xi\theta_x = 0 & \text{in} \quad Q, \\
\rho h v_{tt} - k\left(u + v_x\right)_x + \mu\theta_{xx} = 0 & \text{in} \quad Q, \\
\theta_t - \theta_{xx} = 0 & \text{in} \quad Q, \\
u\left(0, \cdot\right) = 0, \quad u\left(L, \cdot\right) = 0 & \text{in} \quad \left(0, T\right), \\
\mu\theta_x\left(0, \cdot\right) - k v_x\left(0, \cdot\right) = m\left(t\right), \quad v_x\left(L, \cdot\right) = 0 & \text{in} \quad \left(0, T\right), \\
\theta_x\left(0, \cdot\right) = \delta\left(t\right) & \text{in} \quad \left(0, T\right), \\
u\left(\cdot, 0\right) = u_0, \quad u_t\left(\cdot, 0\right) = u_1 & \text{in} \quad \left(0, L\right), \\
v\left(\cdot, 0\right) = v_0, \quad v_t\left(\cdot, 0\right) = v_1 & \text{in} \quad \left(0, L\right), \\
\theta\left(\cdot, 0\right) = \theta_0 & \text{in} \quad \left(0, L\right).
\end{cases}
$$

Thermoelastic system

$$
\begin{cases}
\rho h v_{tt} + v_{xxxx} - \dfrac{\rho h^3}{12} v_{xxtt} + (\xi + \mu)\theta_{xx} = 0 & \text{in} \quad Q, \\
\theta_t - \theta_{xx} = 0 & \text{in} \quad Q, \\
v_x(0,\cdot) = v_x(L,\cdot) = 0 & \text{in} \quad Q, \\
v_{xxx}(0,\cdot) = m(t), \quad v_{xxx}(L,\cdot) = 0 & \text{in} \quad (0,T), \\
\theta_x(0,\cdot) = \delta(t), \quad \theta_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\
v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1 & \text{in} \quad (0,L), \\
\theta(\cdot,0) = \theta_0 & \text{in} \quad (0,L).
\end{cases}
$$

Adjoint system - spectral analysis

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k(\phi + \psi_x) = 0 & \text{in} \quad Q, \\ \rho h \psi_{tt} - k(\phi + \psi_x)_x = 0 & \text{in} \quad Q, \\ -\rho_t - \rho_{xx} - \xi \phi_x + \mu \psi_{xx} = 0 & \text{in} \quad Q, \\ \phi(0,\cdot) = \phi(L,\cdot) = \psi_x(0,\cdot) = \psi_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\ \rho_x(0,\cdot) = \rho_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\ \phi(\cdot,T) = \phi_0, \quad \phi_t(\cdot,T) = \phi_1 & \text{in} \quad (0,L), \\ \psi(\cdot,T) = \psi_0, \quad \psi_t(\cdot,T) = \psi_1 & \text{in} \quad (0,L), \\ \rho(\cdot,T) = \rho_0 & \text{in} \quad (0,L). \end{cases}$$

Solutions for adjoint system

$$\phi(x,t) = \sum_{m\in\mathbb{N}} A_m e^{i\lambda_{km}t} \sin\left(\frac{m\pi x}{L}\right),$$

$$\psi(x,t) = \sum_{m\in\mathbb{N}} \nu_{km} A_m e^{i\lambda_{km}t} \cos\left(\frac{m\pi x}{L}\right)$$

$$\rho(x,t) = \sum_{m\in\mathbb{N}} \left[\gamma_{km} A_m e^{i\lambda_{km}t} + B_m e^{-m^2 t}\right] \cos\left(\frac{m\pi x}{L}\right).$$

Uniform observability

Theorem (Komornik-Tenenbaum (EECT'15))

Let $f = f(t)$ *be of the form* $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} + \sum_{m\in\mathbb{Z}} b_m e^{-\mu_m t}$, *where* $(\lambda_n)_n$ *and* $(\mu_m)_m$ *are sequences of real numbers such that there exist* $\gamma > 0$ *and* $\alpha > 1$ *such that*

$$\gamma := \inf_{n\in\mathbb{Z}} \{\lambda_{n+1} - \lambda_n\} > 0$$

$$\inf_{m\in\mathbb{N}} \{\mu_{m+1} - \mu_m\} > 0, \qquad \sum_{|\mu_m - \mu|\leq t} 1 \ll t^{1/\alpha} \quad (\mu_m, \mu > 0, \quad t \geq 1).$$

Let $T > 0$ *be such that* $T > 2\pi/\gamma$. *Then*

$$\sum_{n\in\mathbb{Z}} |a_n|^2 + \sum_{m\in\mathbb{N}} |b_m|^2 e^{-\mu_m T} \ll \int_0^T |f(t)|^2 \, dt.$$

Uniform observability

Theorem

Let $f = f(t)$ *be of the form* $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} + \sum_{m\in\mathbb{Z}} b_m e^{-\mu_m t}$, *where* $(\lambda_n)_n$ *and* $(\mu_m)_m$ *are sequences of real numbers such that there exist* $\gamma > 0$ *and* $\alpha > 1$ *such that*

$$\gamma_\infty := \inf_{|n|>N} \{\lambda_{n+1} - \lambda_n\} > 0, \qquad \gamma := \inf_{n\in\mathbb{Z}} \{\lambda_{n+1} - \lambda_n\} > 0,$$

$$\inf_{m\in\mathbb{N}} \{\mu_{m+1} - \mu_m\} > 0, \qquad \sum_{|\mu_m - \mu|\le t} 1 \ll t^{1/\alpha} \quad (\mu_m, \mu > 0, \quad t \ge 1).$$

Let $T > 0$ *be such that* $T > 2\pi/\gamma_\infty$. *Then*

$$\sum_{n\in\mathbb{Z}} |a_n|^2 + \sum_{m\in\mathbb{Z}} |b_m|^2 e^{-\mu_m T} \ll \int_0^T |f(t)|^2 \, dt.$$

Uniform observability

Theorem (A, Mercado, de Teresa (to appear))

Let $T > 2L\sqrt{\rho h^3/12}$. Then, there exists a constant $C = C(T)$ such that,

$$\|(\phi(T), \phi_t(T), \psi(T), \psi_t(T), \rho(T))\|^2_{\mathcal{X}\times L^2(0,L)} \le C\left(\int_0^T |\psi_t(0,t)|^2 + |\rho(0,t)|^2 dt\right).$$

Thank you very much for your attention!