The Calderón problem with corrupted data

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Outline

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The inverse Calderón problem

The **inverse Calderón problem** aims at determining the conductivity of an inhomogeneous conductive medium from non-invasive measurements.

The formulation

If f is an **electric potential** prescribed on ∂D, the **electric potential** u inside of D satisfies

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } D, \\ u|_{\partial D} = f. \end{cases}$$

- $\gamma \partial_\nu u|_{\partial D}$ is the **outgoing electric current** density through ∂D.
- Measurements: the **Dirichlet-to-Neumann map**

$$\Lambda_\gamma : f \longmapsto \gamma \partial_\nu u|_{\partial D}$$

The **inverse Calderón problem** is

- to decide if γ is uniquely determined by Λ_γ,
- and to calculate γ in terms of Λ_γ if γ is determined by Λ_γ.

Discussing the model

This problem originates as a theoretical model in electrical prospecting.

- The aim is to determine the conductivity conductivity by means of steady state electrical measurements on ∂D.

Ideally, Λ_γ is determined through measurements effected on ∂D.
The model assumes to have access (to the graph of the DN map):

- to infinite many pieces of data
- and to infinite-precision measurements.

This is unjustified (data do not lie on the graph of the DN map):

- only a finite number of measurements are available
- the data is corrupted by measurement errors

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Boundary reconstruction (smooth setting)

In 1988 Sylvester–Uhlmann show that, whenever D and γ are smooth, the DN map Λ_γ can be locally identified with a first order pseudodifferential operator, and its symbol can be expanded as:

$$\sum_{j=0}^{\infty} \partial_\nu^j \gamma.$$

Then, to reconstruct $\partial_\nu^j \gamma|_{\partial D}$ we only need to recover the symbol of a pseudodifferential operator —this is well known.

$$P = \sum_{|\alpha| \le m} a_\alpha(x)(-i)^{|\alpha|}\partial_x^\alpha \quad \Rightarrow \quad e^{-ix\cdot\xi} P(e^{ix\cdot\xi}) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$$

The **plane waves** $e^{ix\cdot\xi}$ are the tools.

Boundary reconstruction (non-regular setting)

In 2001 Brown used solutions with highly oscillatory Dirichlet data concentrating around a point $P \in \partial D$ to recover

$$\gamma(P).$$

To visualize these solutions think of $P = 0 \in \partial D$ and

$$D \subset \{x \in \mathbb{R}^d : x_d > 0\}.$$

Then, the Dirichlet data of the solutions looks like

$$M^{(d-1)/2} N^{-1/2} \chi(Mx) e^{N(i\xi - e_d)\cdot x}.$$

In 2006 Brown–Salo modified the method to recover:

$$\partial_\nu \gamma(P).$$

The tools are **wave packets**

$$f_{t,\lambda}(x) = t^{d/2} \chi(t(x - x_0)) e^{it^\lambda (x-x_0)\cdot \xi_0}.$$

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The Calderón problem with noisy data

Recall that the goal is to reconstruct γ from the DN map

$$\int_{\partial D} \Lambda_\gamma f \, g \; = \int_{\partial D} \gamma \partial_\nu u \, g$$

where

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } D, \\ u|_{\partial D} = f. \end{cases}$$

In order to avoid the infinite-precision assumption of Calderón's formulation, we assume data to be the DN map plus a random error:

$$\mathcal{N}_\gamma(f, g) = \int_{\partial D} \Lambda_\gamma f \, g \; + \mathcal{E}(f, g),$$

where we want $\mathcal{E}(f, g)$ to denote a centred complex Gaussian whose variance depends on f and g.

Comments on the expectation

Note that

$$\mathbb{E}\mathcal{N}_\gamma(f,g) = \int_{\partial D} \Lambda_\gamma f\, g.$$

Therefore, the noise can be filtered having access to many independent outcomes:

$$\frac{1}{N}\sum_{n=1}^{N} \mathcal{N}_\gamma(f,g)_{(\omega_n)} \xrightarrow[N\to\infty]{} \int_{\partial D} \Lambda_\gamma f\, g.$$

- A few repetitions of the same measurement do not oscillate enough to filter out the noise by averaging.
- We want to avoid averaging and show that a single realization of $\mathcal{N}_\gamma$ is enough to reconstruct γ.

Comments on the variance

The variance

$$\mathbb{E}\left|\mathcal{N}_\gamma(f,g) - \int_{\partial D} \Lambda_\gamma f\, g\right|^2$$

depends on f and g.

- The *variable* variance aims at modelling measurement devices which decalibrates as the *strength* of the electric potential and the outgoing current increases.

Different approaches

There seem to be two different approaches.

- Deterministic regularization [Tikhonov]: The noise is deterministic and small.
- Statistical point of view: [Sudakov–Halfin, Franklin] No smallness assumption for the noise. [Abraham–Nickl] The level of noise is small.

Our approach is stochastic with no restriction on the size of the noise. It seems to be a new approach for the Calderón problem.

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Reconstruction of $\gamma|_{\partial D}$

Theorem (C, Garcia)

For every $P \in \partial D$, there exists an explicit sequence $\{f_N : N \in \mathbb{N}\}$ such that

$$\lim_{N\to\infty} \mathcal{N}_\gamma(f_N, \overline{f_N}) = \gamma(P)$$

almost surely.

Rate of convergence for $\gamma|_{\partial D}$

Theorem (C, Garcia)

For every $P \in \partial D$, there exist an explicit sequence $\{f_N : N \in \mathbb{N}\}$ and a constant $C > 0$ such that, for every $\epsilon > 0$, we have

$$\mathbb{P}\{|\mathcal{N}_\gamma(f_N, \overline{f_N}) - \gamma(P)| \leq CN^{-2/3}\} \geq 1 - \epsilon \qquad \forall\, N \geq c\epsilon^{-3}.$$

Reconstruction of $\partial_\nu \gamma|_{\partial D}$

The DN map of the reference medium with conductivity identically one is denoted by Λ.

Theorem (C, Garcia)

For every $P \in \partial D$, there exists an explicit family $\{f_t : t \geq 1\}$ such that, if

$$Y_N = \frac{1}{N^4} \int_{N^4}^{2N^4} \left[\mathcal{N}_\gamma(f_{t^2}, \overline{f_{t^2}}/\gamma) - \int_{\partial D} \Lambda f_{t^2}\, \overline{f_{t^2}} \right] dt,$$

one has that

$$\lim_{N \to \infty} Y_N = \frac{\partial_{\nu_P} \gamma(P) + i\tau_P \cdot \nabla \gamma(P)}{\gamma(P)}$$

almost surely. Here ν_P is the outward unit normal vector to ∂D at P and τ_P denotes any unitary tangential vector at P.

Why the need of Y_N?

Recall that Λ_γ is a first order pseudodifferential operator with symbol

$$\sum_{j=0}^{\infty} \partial_\nu^j \gamma.$$

Since $\mathcal{N}_\gamma(f_{t^2}, \overline{f_{t^2}}) \to \gamma(P)$, then

$$\mathcal{N}_\gamma(f_{t^2}, \overline{f_{t^2}}/\gamma) \to 1.$$

Consequently,

$$\mathcal{N}_\gamma(f_{t^2}, \overline{f_{t^2}}/\gamma) - \int_{\partial D} \Lambda f_{t^2}\, \overline{f_{t^2}} \to \frac{\partial_{\nu_P}\gamma(P)}{\gamma(P)}.$$

Rate of convergence for $\partial_\nu \gamma|_{\partial D}$

Theorem (C, Garcia)

Consider $P \in \partial D$ and $\{f_t : t \geq 1\}$ the family of the previous theorem. For every $N \in \mathbb{N}$, set

$$Y_N = \frac{1}{N^4} \int_{N^4}^{2N^4} \left[\mathcal{N}_\gamma(f_{t^2}, \overline{f_{t^2}}/\gamma) - \int_{\partial D} \Lambda f_{t^2}\, \overline{f_{t^2}} \right] dt.$$

Then, there exists a constant $C > 0$ such that, for every $\epsilon > 0$, we have

$$\mathbb{P}\left\{ \left| Y_N - \frac{\partial_{\nu_P}\gamma(P) + i\tau_P \cdot \nabla\gamma(P)}{\gamma(P)} \right| \leq CN^{-2/3} \right\} \geq 1 - \epsilon \qquad \forall\, N > c\epsilon^{-3}.$$

Comments on the results

- The main contributions are to filter the measurement errors almost surely and provide a *highly probable* rate of convergence.
- For the first two theorems no average is needed because

$$\|f_N\|_{L^2(\partial D)} = \mathcal{O}(N^{-1/2}).$$

- For the other two theorems we require an average in $\sqrt{N}$ since

$$\|f_N\|_{L^2(\partial D)} = \mathcal{O}(1).$$

The strong law of large numbers

The underlying principle is the Strong Law of Large Numbers:
If $\{X_n\}$ is a sequence of independent random variables with means $\{\mu_n\}$ and variances $\{\sigma_n^2\}$ such that

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} < \infty,$$

then

$$\frac{1}{N}\sum_{n=1}^{N}(X_n - \mu_n) \to 0$$

almost surely.
Recall

$$\mathcal{N}_\gamma(f,g) = \int_{\partial D} \Lambda_\gamma f\, g \; + \mathcal{E}(f,g).$$

The idea is to see that there is enough uncorrelation for

$$\mathcal{E}(f_t, \overline{f_t}).$$

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White noise

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{H}$ be a Hilbert space. A linear map $\mathbb{W} : \mathcal{H} \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a **complex Gaussian white noise** if, for every $f \in \mathcal{H}$, $\mathbb{W}f$ is a centred complex Gaussian variable and

$$\mathbb{E}(\overline{\mathbb{W}f}\,\mathbb{W}g) = (f|g)_{\mathcal{H}}, \text{ for all } f, g \in \mathcal{H}.$$

Here $(.|.)_{\mathcal{H}}$ denotes the inner product of $\mathcal{H}$.

Existence of white noise

If $\mathcal{H}$ is a separable Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space so that there exists a sequence $\{X_n : n \in \mathbb{N}\}$ of independent complex Gaussian so that

$$\mathbb{E}X_n = 0, \qquad \mathbb{E}(\overline{X_n}X_n) = 1,$$

then, there always exists a complex Gaussian white noise for $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{H}$.
It is enough to consider an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$, and define

$$\mathbb{W}f = \sum_{n\in\mathbb{N}} (e_n|f)_{\mathcal{H}} X_n,$$

where the convergence takes place in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

The model for the error

We define the error $\mathcal{E}$ from a complex Gaussian white noise $\mathbb{W}$ in the Hilbert space $\mathcal{H} = L^2(\partial D) \otimes L^2(\partial D)$ as follows:

$$\mathcal{E}(f, g) = \mathbb{W}(f \otimes g).$$

Recall that

$$(f_1 \otimes g_1 | f_2 \otimes g_2)_{\mathcal{H}} = (f_1|f_2)(g_1|g_2),$$

with $(\phi|\psi) = \int_{\partial D} \phi \overline{\psi}$.

This makes the error $\mathcal{E}$ take the form

$$\mathcal{N}_\gamma(f, g) = \int_{\partial D} \Lambda_\gamma f \, g \; + \sum_{\alpha \in \mathbb{N}^2} (f|e_{\alpha_1})(g|e_{\alpha_2}) X_\alpha$$

with $X_\alpha = \mathbb{W}(e_{\alpha_1} \otimes e_{\alpha_2})$ and $\{e_n : n \in \mathbb{N}\}$ an o.n.b. of $L^2(\partial D)$.

- $\mathcal{N}_\gamma(f, g)$ is complex Gaussian
- $\mathbb{E}\mathcal{N}_\gamma(f, g) = \int_{\partial D} \Lambda_\gamma f \, g$
- $\mathbb{E}|\mathcal{N}_\gamma(f, g) - \int_{\partial D} \Lambda_\gamma f \, g|^2 = \|f\|^2_{L^2(\partial D)} \|g\|^2_{L^2(\partial D)}$

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Maxwell's equations

In collaboration with Lai, Lin and Zhou, we extended the results for the time-harmonic Maxwell system:

$$\nabla \times E - i\omega\mu H = 0 \text{ in } D,$$
$$\nabla \times H + i\omega\gamma E = 0 \text{ in } D.$$

In this situation the boundary data is given by the map:

$$\nu \times E|_{\partial D} \mapsto \nu \times H|_{\partial D}.$$

We reconstructed $\mu|_{\partial D}$ and $\gamma|_{\partial D}$ from noisy data modelling the error in

$$\mathcal{H} = L^2(\partial D)^3 \otimes L^2(\partial D)^3 \quad \text{and} \quad \mathcal{H} = H^{-1}(\partial D)^3 \otimes H^{-1}(\partial D)^3.$$

- Need of averaging to reconstruct $\mu|_{\partial D}$ and $\gamma|_{\partial D}$ when the error is modelled in $\mathcal{H} = L^2(\partial D)^3 \otimes L^2(\partial D)^3$.
- No need of averaging when the error is modelled in $\mathcal{H} = H^{-1}(\partial D)^3 \otimes H^{-1}(\partial D)^3$.

The problem of observability with noise

To understand the role of the choice of $\mathcal{H}$ in the error $\mathcal{E}$, we introduce the problem of recovering an **observable** P from certain **measurements** $\mathcal{N}_P$ that contain some random errors.

- The observable P is a pseudodifferntial operator

$$Pf(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\widehat{f}(\xi)\,\mathrm{d}\xi,$$

  with a classical symbol a of order $m \in \mathbb{R}$.
- The measurements

$$\mathcal{N}_P(f,g) = \int_{\mathbb{R}^d} \overline{f} Pg + \mathcal{E}_\beta(\overline{f}, g),$$

  with $\mathcal{H} = H^\beta(\mathbb{R}^d) \otimes H^\beta(\mathbb{R}^d)$.

Different βs correspond to different variances of $\mathcal{E}_\beta$.

- If $\beta = 0$ the oscillations of f and g do not affect the size of the error, only their masses.
- For $\beta > 0$ the noise increase with the oscillations of f and g.
- For $\beta < 0$ the oscillations reduce the size of the noise.

The observational limit of wave packets

Assume

$$a \sim \sum_{j=1}^{\infty} a_j,$$

with a_j being a classical symbol of order $m_j \in \mathbb{R}$, for $m_j < m_{j-1} < \cdots < m_1 = m$, which is *homogeneous in the variable* ξ.
In collaboration with Meroño, showed how to use wave packets to reconstruct

$$a_1, \ldots, a_{j_\beta}, a_{j_\beta+1}, \ldots, a_{k_\beta}$$

when the observable P is so that

$$m = m_1 > \cdots > m_{j_\beta} > 2\beta \geq m_{j_\beta} > \cdots > m_{k_\beta} > 2\beta - 1/2.$$

- For $a_1, \ldots, a_{j_\beta}$ no averaging is needed.
- For $a_{j_\beta+1}, \ldots, a_{k_\beta}$ averaging is required.

Furthermore, it is not possible to use wave packets to reconstruct

$$a_{k_\beta+1}, a_{k_\beta+2}, \ldots$$

in presence of the error. The signal is lost in the noise.

A particular case to keep in mind

- If the observable P is a differential operator of order m, we can recover the full operator P from $\mathcal{N}_{\beta,P}$ with $\beta < 1/4$.
- Even with an error $\mathcal{E}_\beta$ that gets amplified with oscillations, one can obtain the full observable.

Outline

To take home

- The Calderón problem is a theoretical model that arises in electrical prospecting.
- Implementing the model presents non-trivial challenges since it assumes infinite-precision measurements and infinite many pieces of data.
- We consider the problem of data corruption in the boundary reconstruction. Our approach is stochastic and provides reconstruction almost surely.
- We also show a *highly probable* rate of convergence.
- The key underlying principle is the Strong Law of Large Numbers.
- The results can be extended to Maxwell's equations.
- There is a freedom in the choice of modelling the error. Different choices respond to oscillations in very different ways.
- Wave packets are useful but have limitations in the problem of observability with noise.