# Control and inverse problems under fully non-linear PDE systems. Two examples.

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- Too ambitious: even direct problems are not yet understood
- Methods for finding solutions for direct problems? The only technique, for non-perturbed situations, is variational: one tries to find solutions of problems by showing existence of minimizers for associated functionals
- Fundamental difficulty: vector nature of functionals. Convexity can very rarely be a reasonable assumption in relevant situations.

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- In practice, direct problems amount to solving (approximating) equilibrium system of PDEs

$$\operatorname{div}\left(W_{\mathbf{F}}(\nabla\mathbf{v})\right) = \mathbf{0} \text{ in } \Omega, \quad \text{boundary conditions on } \partial\Omega.$$

# Non-linear system of PDEs

Equilibrium equations for direct problem: feasible solution $\mathbf{u}$ with

$$\int_\Omega W_{\mathbf{F}}(\nabla \mathbf{u}) : \nabla \mathbf{v} \, d\mathbf{x} = 0$$

for every test $\mathbf{v}$ with $\mathbf{v} = \mathbf{0}$ on $\Gamma_D$. This is the weak form of

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- But $W$ is not quadratic nor convex !!! System of PDEs is non-linear, non-elliptic (in the classical sense).
- Existence of solutions is achieved through polyconvexity (Ball, 1978, Ciarlet, 1988), and the remarkable properties of minors. Uniqueness does not hold in general.

# Minors and polyconvexity

Fundamental property of minors:

$$\nabla \mathbf{u}_j \rightharpoonup \nabla \mathbf{u} : \quad \operatorname{cof} \nabla \mathbf{u}_j \rightharpoonup \operatorname{cof} \nabla \mathbf{u}, \det \nabla \mathbf{u}_j \rightharpoonup \det \nabla \mathbf{u}$$

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Polyconvexity:

$$W(\mathbf{F}) = g(\mathbf{F}, \operatorname{cof} \mathbf{F}, \det \mathbf{F}) : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R} \to \mathbb{R}, \quad g, \text{ convex.}$$

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It is shown that, under polyconvexity, the variational problem

$$\text{Minimize in } \mathbf{u} \in \mathcal{A} : \quad E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x},$$
$$\mathcal{A} = \{\mathbf{u} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{u} = \mathbf{0} \text{ in } \Gamma_D\},$$

admits, at least, one minimizer: a solution of above system.

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- control: inner pressure $t(\mathbf{x})$ acting on every point in $\Omega$;
- objective: given two fields $\mathbf{s}(\mathbf{x})$ and $\mathbf{s}_{des}(\mathbf{x})$ (the direction of the stem and the direction indicating the sun, respectively), choose the inner pressure $t(\mathbf{x})$ so as to align, as closely as possible, $\mathbf{s}(\mathbf{x})$ with $\mathbf{s}_{des}(\mathbf{x})$.

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$$\text{Min in } t: \quad \frac{1}{2}\int_\Omega \|(\mathbf{id}+\nabla\mathbf{u})\mathbf{s}-\mathbf{s}_{des}\|^2\,d\mathbf{x}+\frac{M}{2}\int_\Omega t^2\,d\mathbf{x}+\frac{\epsilon}{2}\int_\Omega |\nabla t|^2\,d\mathbf{x},$$

subject to

$$\int_\Omega \nabla_{\mathbf{F}} W(\mathbf{id}+\nabla\mathbf{u}(\mathbf{x})) : \nabla\mathbf{v}(\mathbf{x})\,d\mathbf{x} = \int_\Omega t(\mathbf{x})\,\mathrm{cof}(\mathbf{id}+\nabla\mathbf{u}(\mathbf{x})) : \nabla\mathbf{v}(\mathbf{x})\,d\mathbf{x}$$

for all $\mathbf{v}$ in

$$\mathbb{V}_D := \left\{\mathbf{v}\in H^1\left(\Omega;\mathbb{R}^N\right) \,:\, \mathbf{v}=0 \text{ on } \Gamma_D\right\}.$$

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$$\{t_j\},\ \text{minimizing}: \quad \{\mathbf{u}_j\},$$

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Take limits in $j$:

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End of proof: standard weak lower semicontinuity through convexity.

# Approximation, and simulation

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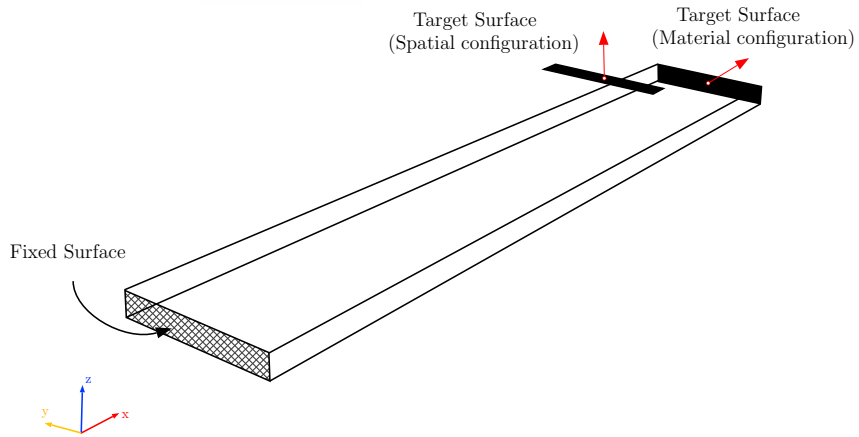

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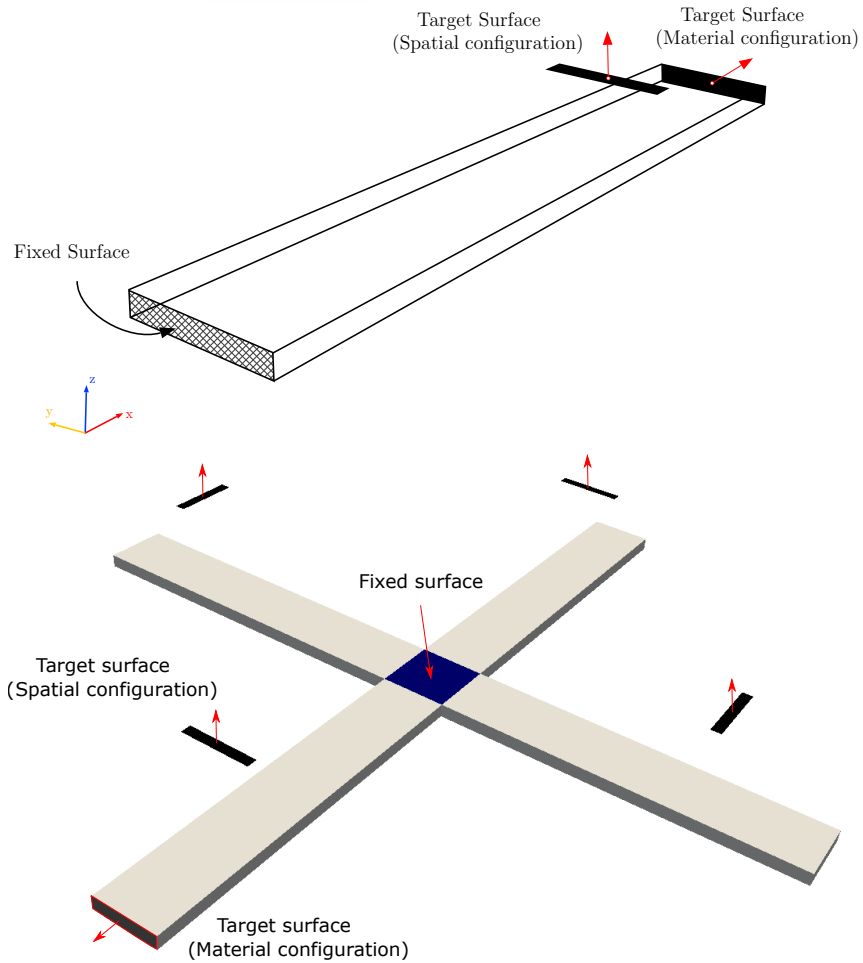

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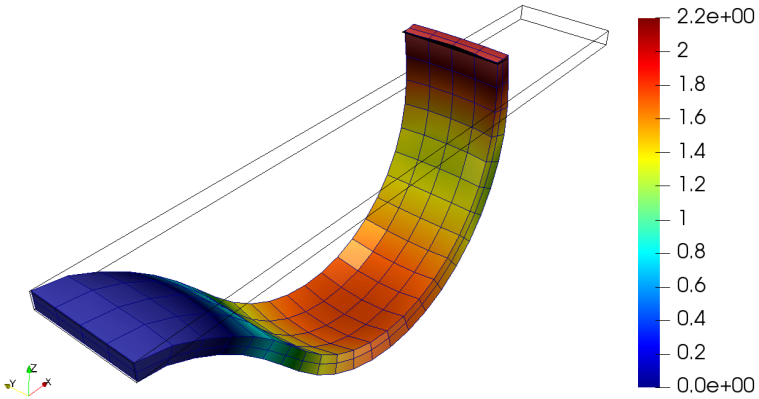

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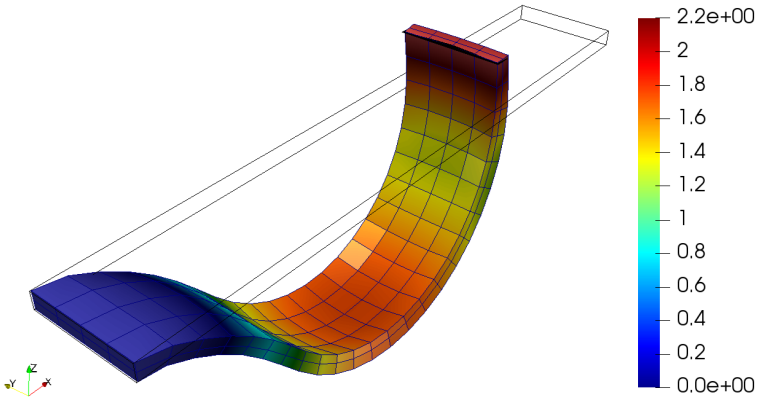


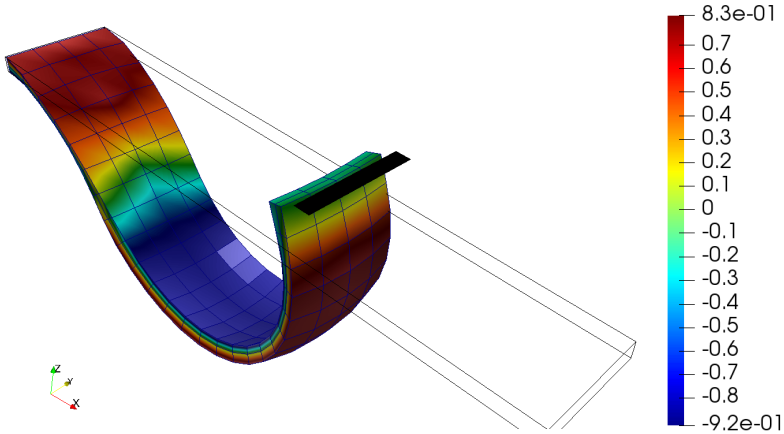

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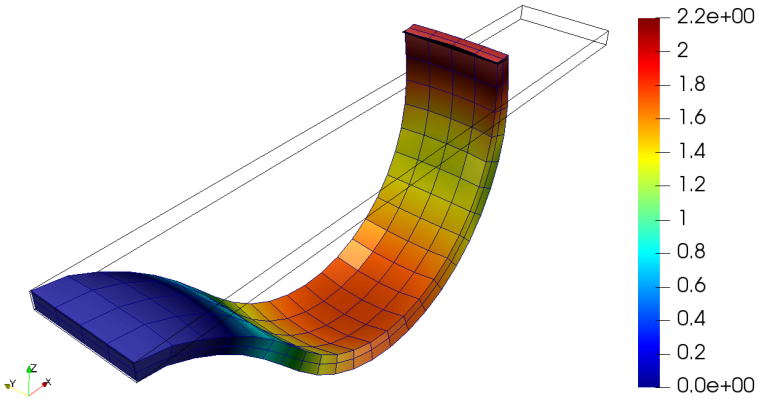


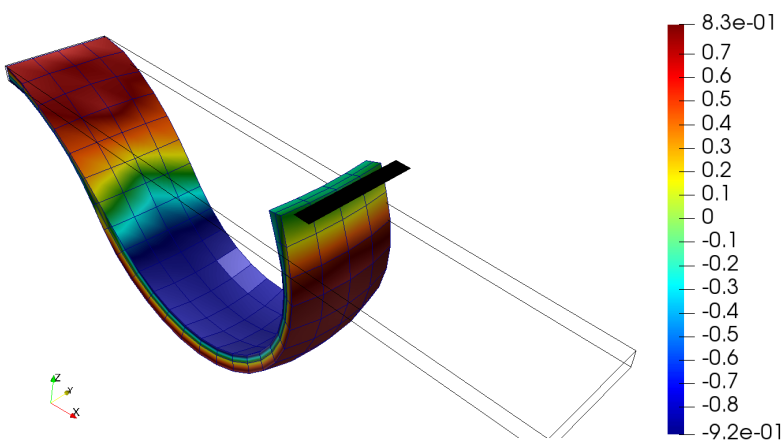


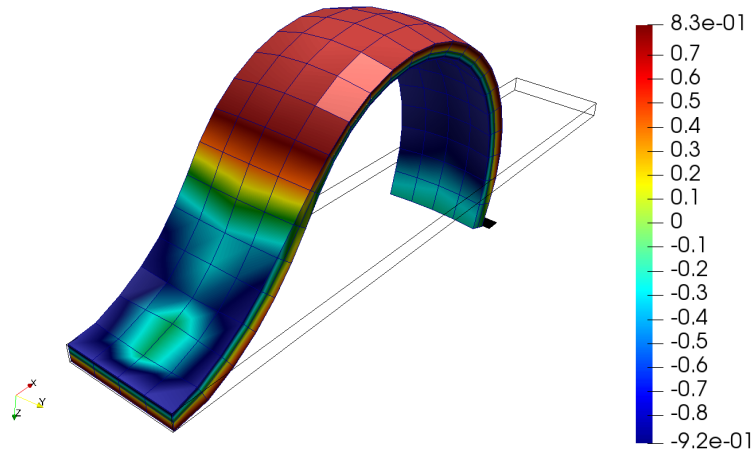

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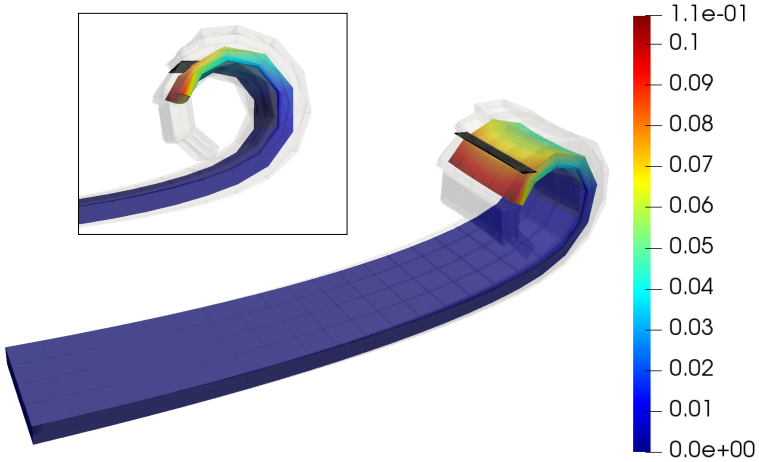

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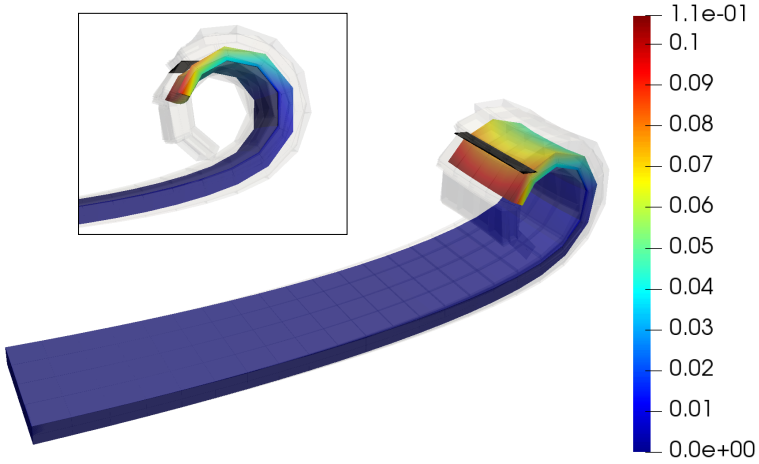


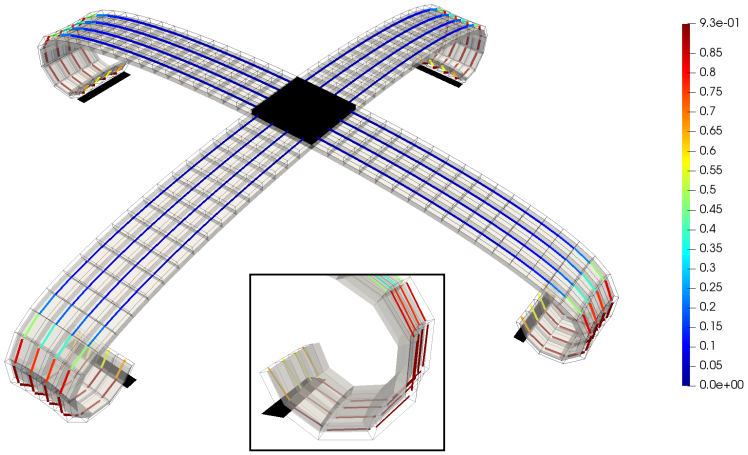

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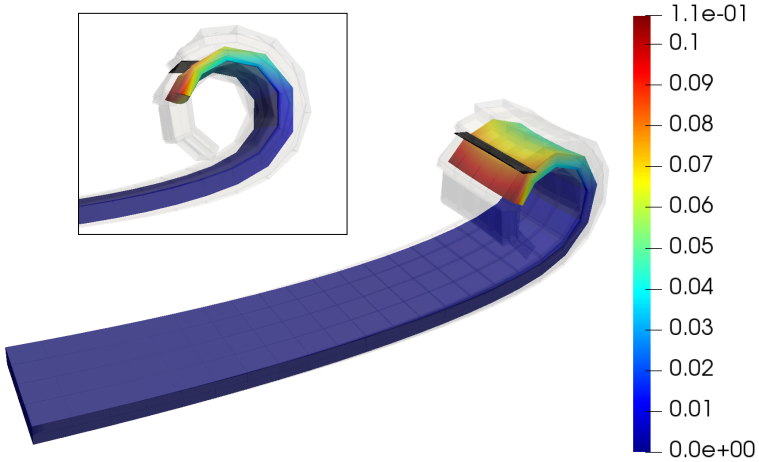


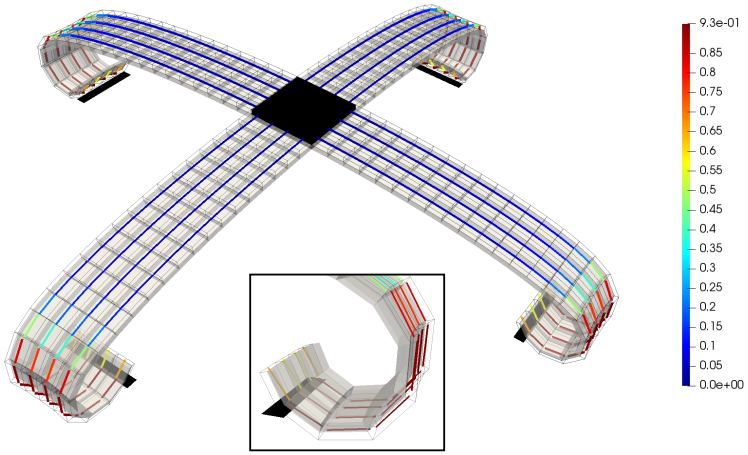


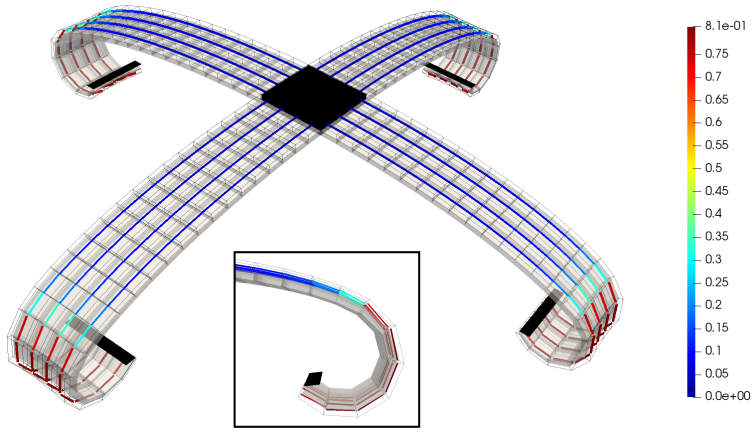

# Second situation. Inverse problems in conductivity

## Basic fact (Beltrami equation)

$$\operatorname{div}(\gamma(\mathbf{x})\nabla u(\mathbf{x})) = 0 \text{ in } \Omega \subset \mathbb{R}^2 \iff \gamma\nabla u + \mathbf{R}\nabla v = \mathbf{0}, \quad \mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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Three main consequences:

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## Strategy

If $\gamma$ is unknown, can try to recover it through pair $(u, v)$, solution of

$$\operatorname{div}\left(\frac{|\nabla v|}{|\nabla u|}\nabla u\right) = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega,$$

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# Underlying functional

System of PDEs: Euler-Lagrange system of functional

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## Elementary fact

Under $\gamma \nabla u + \mathbf{R}\nabla v = \mathbf{0}$:

$$\phi(\nabla u, \nabla v) = |\nabla u|\,|\nabla v| = \gamma |\nabla u|^2 = -\nabla u \cdot \mathbf{R}\nabla v = \det(\nabla u, \nabla v).$$

# New variational problem

$$E(\mathbf{u}) = \int_\Omega \psi(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \quad \psi(\mathbf{F}) = \phi(\mathbf{F}) - \det \mathbf{F}.$$

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## Question

Can one devise some situations in which our non-linear system admits solutions in spite of these difficulties?

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4. Take $v_0 = v|_{\partial\Omega}$.

# Result

System

$$\operatorname{div}\left(\frac{|\nabla v|}{|\nabla u|}\nabla u\right) = 0, \operatorname{div}\left(\frac{|\nabla u|}{|\nabla v|}\nabla v\right) = 0 \text{ in } \Omega,$$

under $u = u_0$, $v = v_0$ on $\partial\Omega$ admits, at least, one solution, and ...

$$\frac{|\nabla v|}{|\nabla u|}\nabla u + \mathbf{R}\nabla v = \mathbf{0},$$

i.e.

$$\gamma = \frac{|\nabla v|}{|\nabla u|},$$

$$\operatorname{div}(\gamma\nabla u) = 0, \quad \operatorname{div}(\frac{1}{\gamma}\nabla v) = 0.$$

# Multi-measurement situation

System with $2N$-PDEs:

$$\operatorname{div}\left(\frac{|\nabla \mathbf{u}_2|}{|\nabla \mathbf{u}_1|}\nabla u_1^{(j)}\right) = 0 \text{ in } \Omega, \quad u_1^{(j)} = u_{1,0}^{(j)} \text{ on } \partial\Omega,$$

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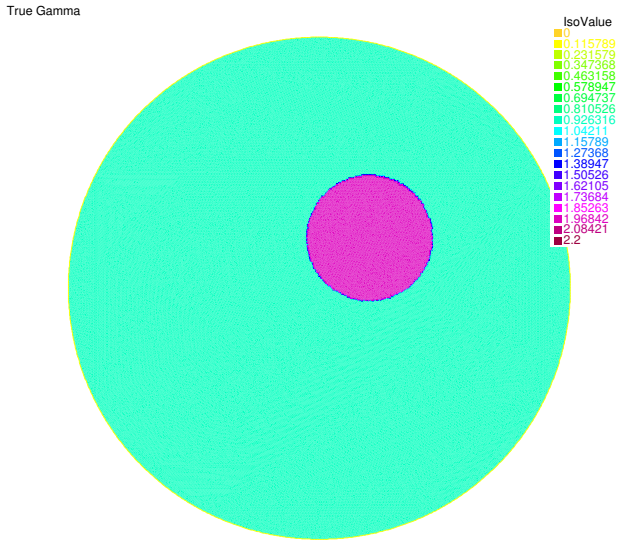
Similar result and parallel method applied componentwise.
Formula for the recovered $\gamma(\mathbf{x})$ in the multimeasurement situation is

$$\gamma(\mathbf{x}) = \frac{|\nabla \mathbf{u}_2(\mathbf{x})|}{|\nabla \mathbf{u}_1(\mathbf{x})|} \quad \text{a.e. } x \in \Omega.$$

# Some simulations



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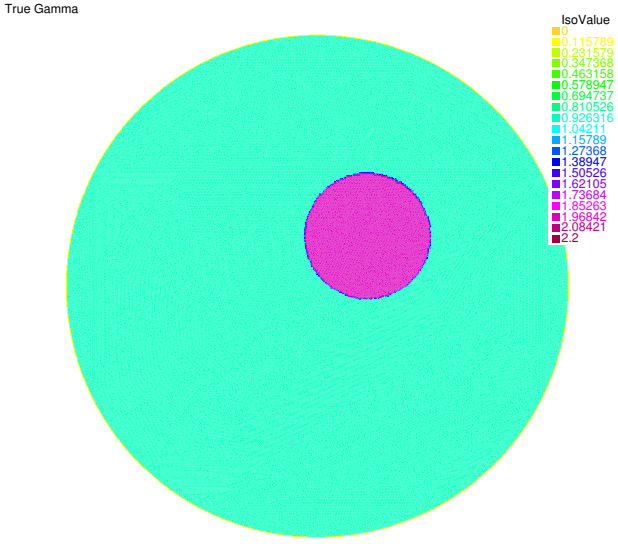


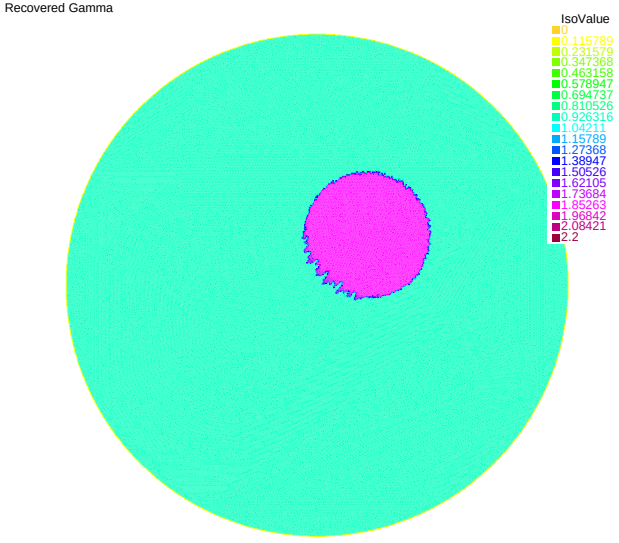

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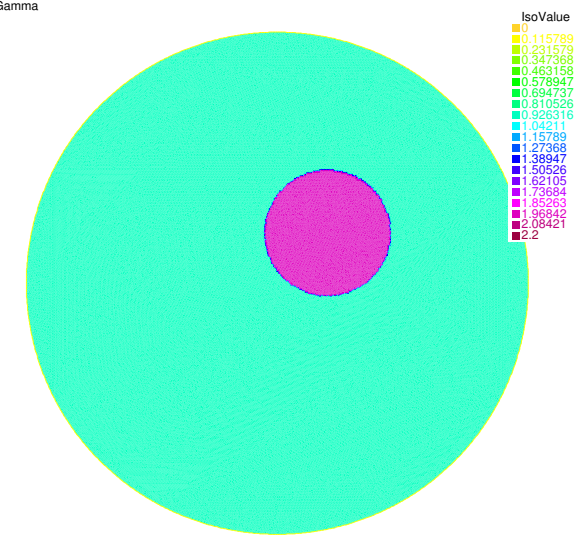
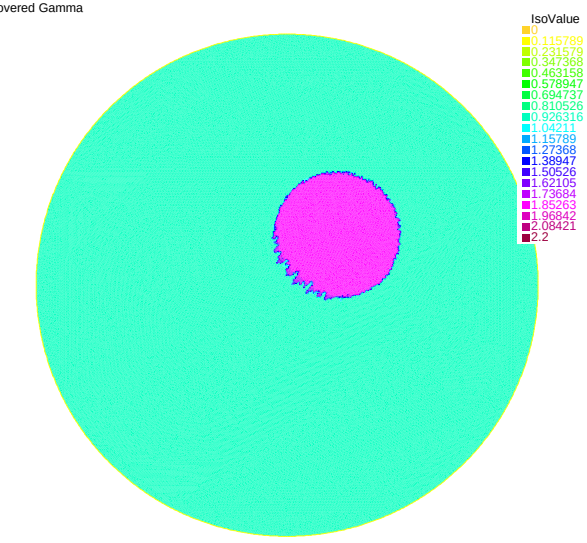
True Gamma




Recovered Gamma

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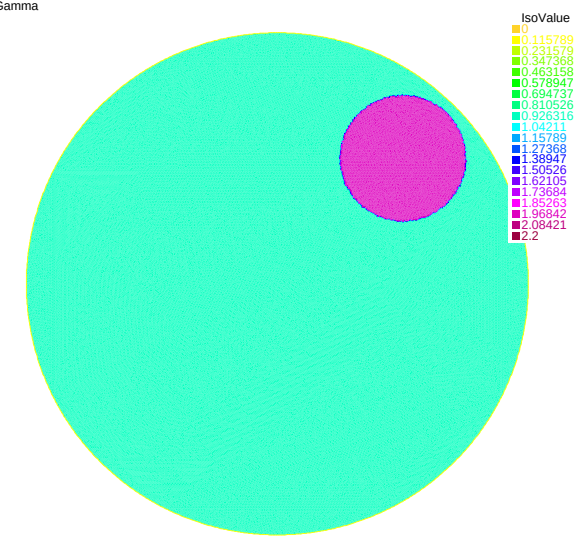

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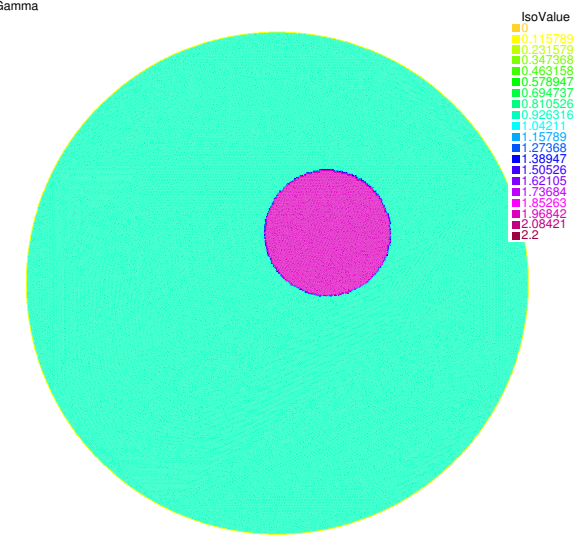


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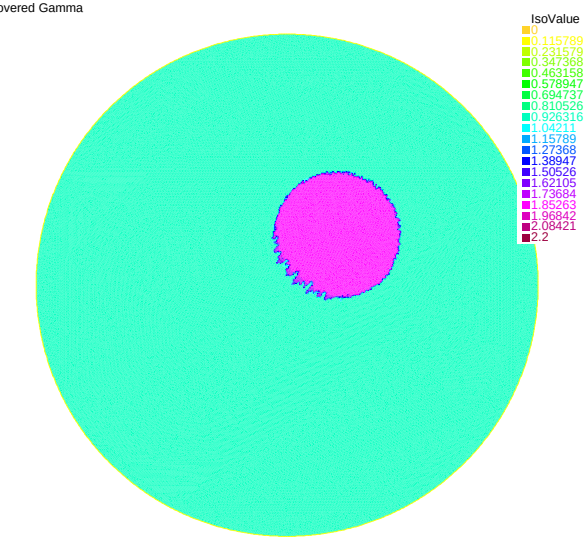


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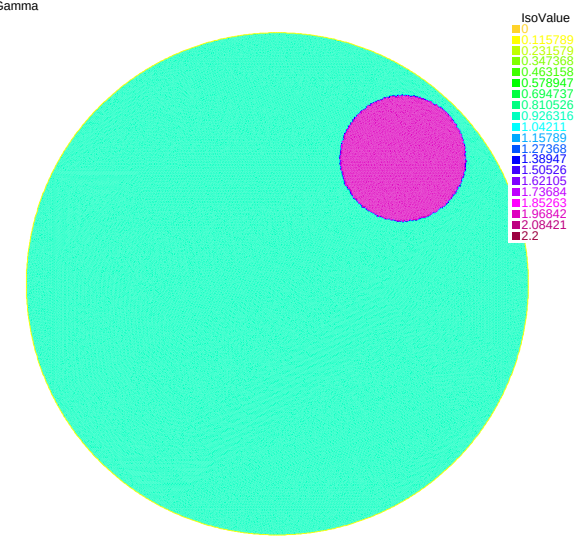


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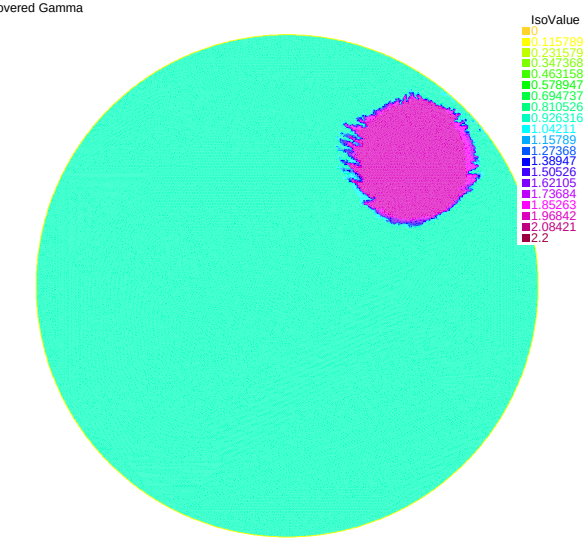

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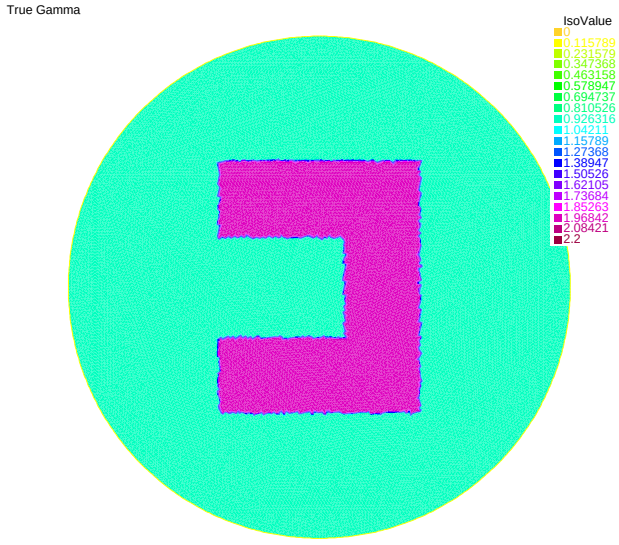

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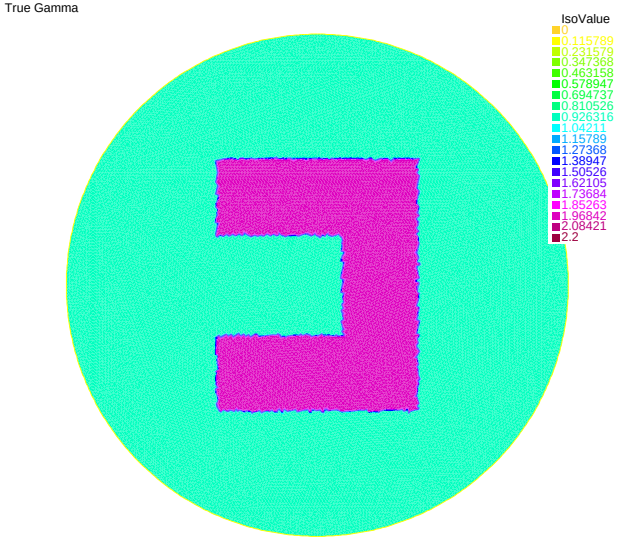


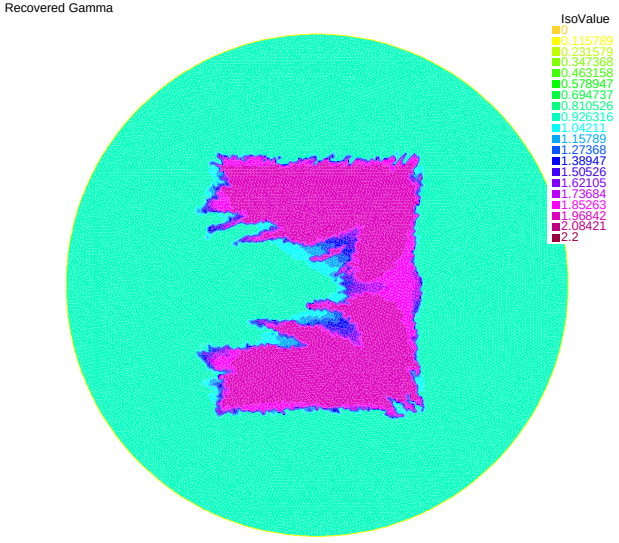

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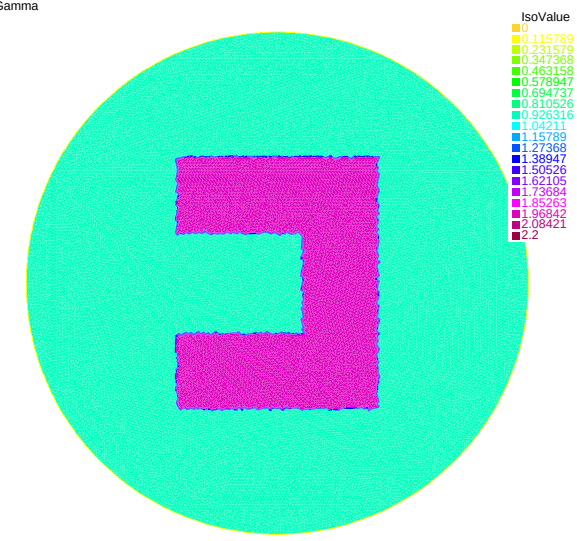


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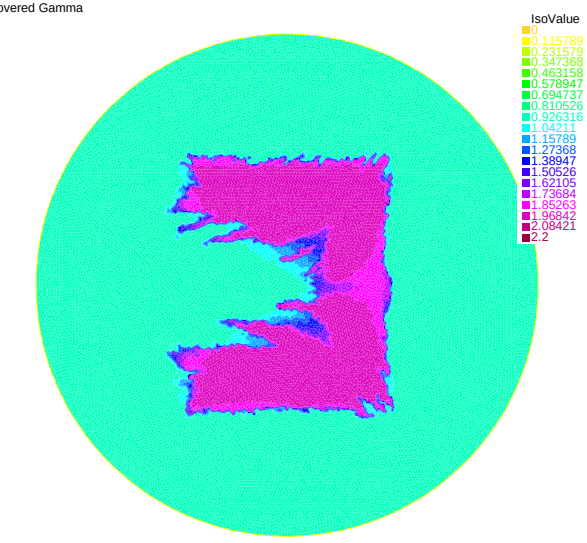


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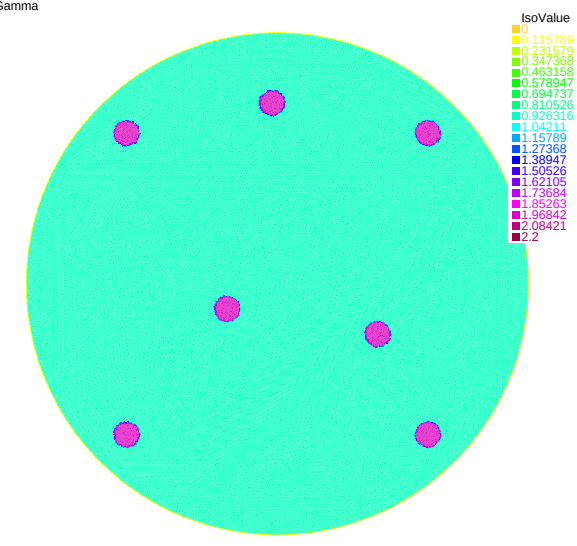

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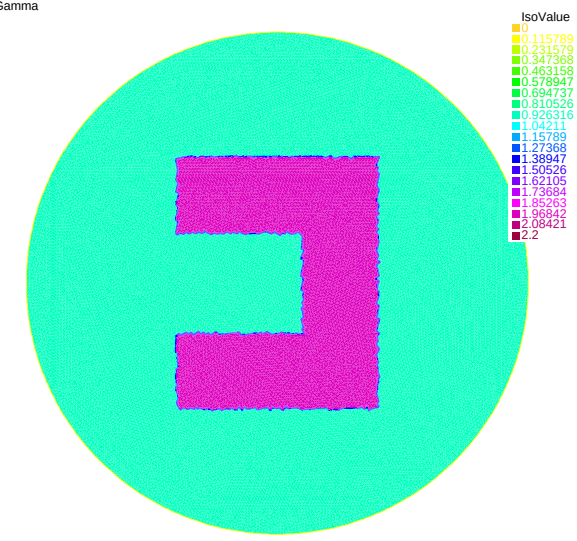


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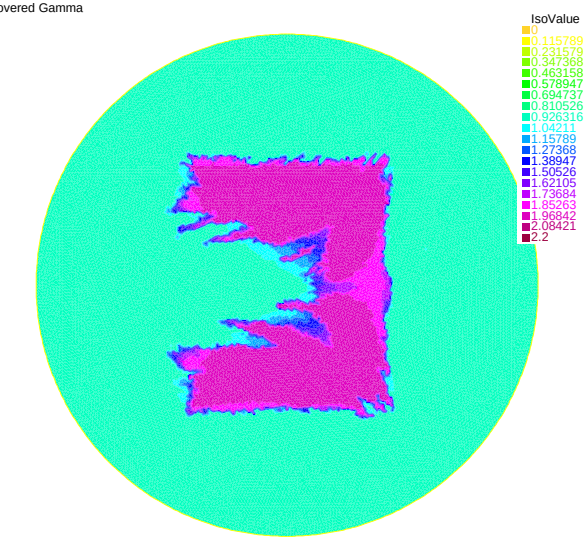


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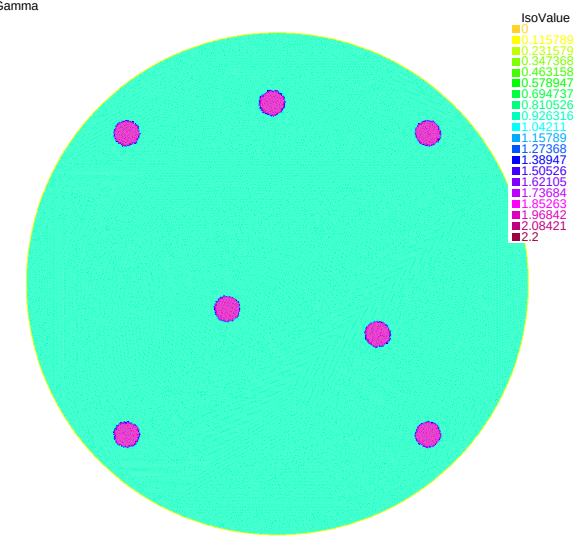

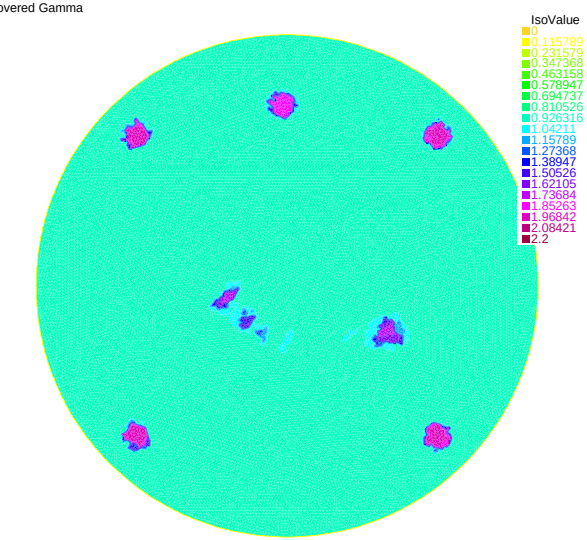
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