Controlling some PDEs with nonlocal terms. Positive and negative results

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Control en Tiempos de Crisis

Wide World, April 2020

Outline

CEMat initiatives and others

http://matematicas.uclm.es/cemat/covid19/
Official data from ISCII (Spain)

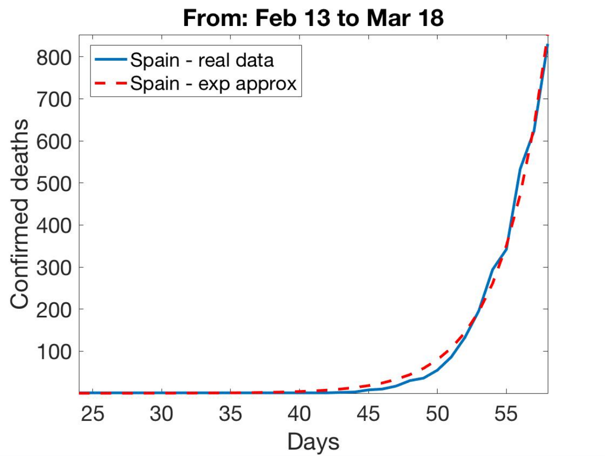
- COVID-19 Cases ? 23/03/2020
- ISCII Data
- Mortality monitoring

http://institucional.us.es/blogimus/

https://www.imperial.ac.uk

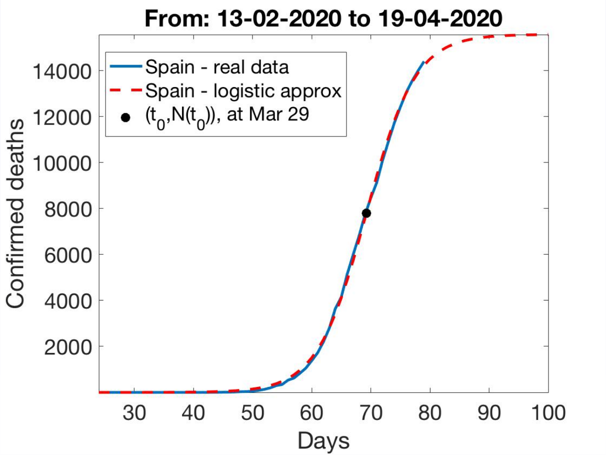
Some computations

Number of deaths - Inital behavior in Spain (exponential)



Some computations

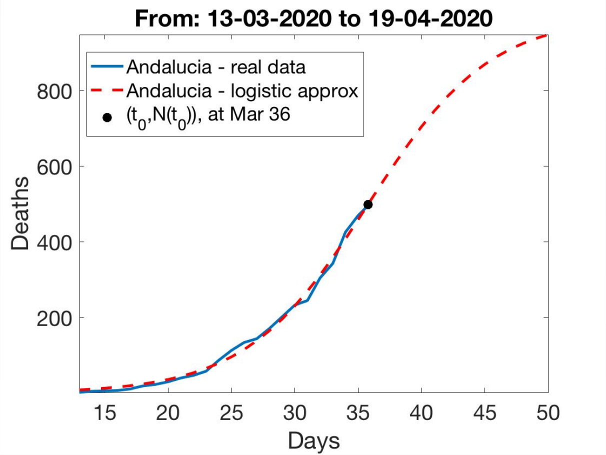
Number of deaths - Present and future behavior in Spain



Estimated final deaths: 15578 Optimistic?

Some computations

Number of deaths - Present and future behavior in Andalucia



Estimated final deaths: 998 Optimistic?

Something very present in our minds:

How can control and/or parameter identification techniques help?

Control oriented to vaccination strategies?

Something very present in our minds:

How can control and/or parameter identification techniques help?

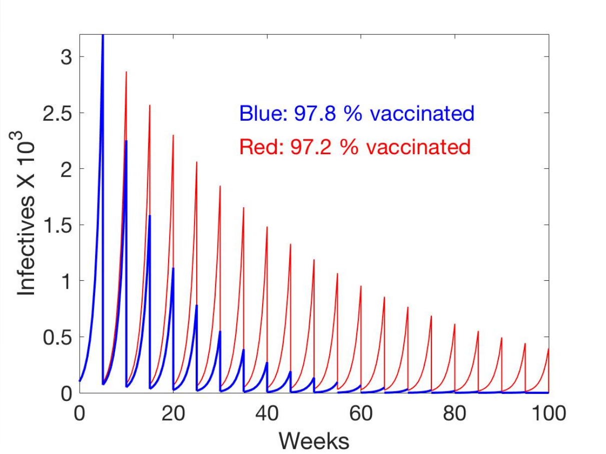


Figure: Results after 5-week periodic actions

Control oriented to vaccination strategies?

CONTROL PROBLEMS FOR ODEs AND PDEs

A general control system for an evolution equation:

$$\begin{cases} y_t + A(y) = Bv \\ + \dots \end{cases}$$

y is the state, $y : [0, T] \mapsto H$
v is the control, $v \in \mathcal{U}$

Two classical approaches:

- Optimal control: Find v such that (v, y) minimizes a cost $J = J(v, y)$
- Controllability: Find v such that $R(y)$ takes a desired value, $R : \mathcal{U} \mapsto \mathcal{Z}$
  For instance, $R(y) := y(T)$

Main questions: Existence? Characterization? Computation?

Control: simple but unsolved

What is preferable: cheap or expensive controls?

The state equation (ODE):

$$x' = f(x) + g(x)u, \quad x \in \mathbf{R}^m, \quad u \in \mathbf{R}^n, \quad n \leq m$$

Cost functionals:

$$J(u) := \int_0^{+\infty} x(t)^T P x(t)\,dt \quad (\rightarrow \text{expensive controls})$$

$$J_\varepsilon(u) = \int_0^{+\infty} x(t)^T P x(t)\,dt + \varepsilon \int_0^{+\infty} u(t)^T R\, u(t)\,dt \quad (\rightarrow \text{cheap controls})$$

where P and R are symmetric, definite positive

An open question [Orlov 2000, Blondel 2014]:

Do we have or not $\inf_u J(u) = \lim_{\varepsilon \to 0} \inf_u J_\varepsilon(u)$?

Motivation

Optimal controls for J (resp. J_ε): expensive (resp. cheap) controls

For instance: in autonomous car driving

$J(\hat{u})$ and $J_\varepsilon(\hat{u}_\varepsilon)$: the associated costs

The answer is yes for linear state systems

Control: simple but unsolved

Minimal time controls for Kepler ODE's:

$$\begin{cases} \text{Minimize } T \\ \text{Subject to } \exists(r,\gamma) \ \text{ satisfying } \ (C) \end{cases}$$

Here

$$\begin{cases} r'' = -k\frac{r}{|r|^3} + \gamma, \quad t \in (0,T) \\ r(0) = r_0, \quad r'(0) = r_1 \\ \gamma \in L^\infty(0,T), \|\gamma\|_\infty \leq \Gamma \\ h(r(T), r'(T)) = 0 \end{cases} \qquad (C)$$

((r_0, r_1) such that uncontrolled $\Rightarrow$ periodic)

Motivation:
Computation of optimal transfer orbits, satelites with low thrust engines (electro-ionic, not chemical, propulsion; low thrust $\sim$ longer transfer time)

Two open questions [Caillau et al 2014]:

- Uniqueness of optimal control?
- Do continuous optimal controls exist?

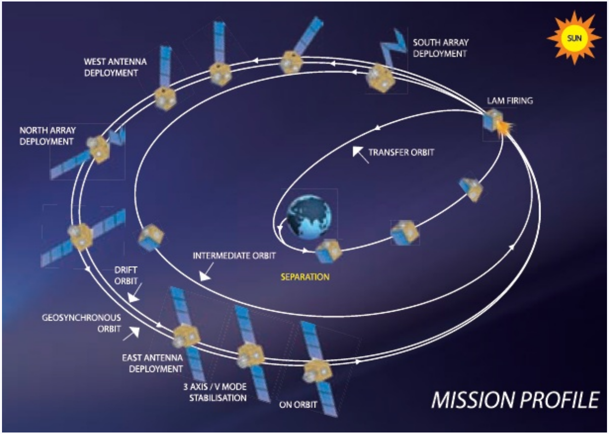


Figure: An orbit transfer for a satelite

Nonlocal in space

Motivation: Nonlocal chemical effects on the temperature, nonlocal behavior of a flux, nonlocal effects of the market in finances, ...

$$\begin{cases} y_t - \Delta y + \int_\Omega K(x,\xi) y(\xi,t)\, d\xi = v 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega \end{cases}$$

NC? $\exists v$ such that $y(x,T) = 0$ in Ω?

Equivalent to observability:

$$\|\varphi(\cdot\,,0)\|^2 \le C \iint_{\omega\times(0,T)} |\varphi|^2 \, dx\, dt \quad \forall \varphi^T \in L^2$$

where

$$\begin{cases} -\varphi_t - \Delta\varphi + \int_\Omega K(\xi,x)\varphi(\xi,t)\, d\xi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^T(x) & \text{in} \quad \Omega \end{cases}$$

Usually: Carleman $\Rightarrow$ Observability - Not here! - We only get:

$$\iint_Q \rho^{-2}|\varphi|^2 \le C_\varepsilon \iint_{\omega\times(0,T)} \rho^{-2}|\varphi|^2 + \varepsilon \iint_Q \rho^{-2} \left| \int_\Omega K(\xi,x)\varphi(\xi,t)\, d\xi \right|^2$$

Theorem (EFC-Li-Zuazua, 2016)

Assume: $K(x,\xi) = \sum_{m,j\geq 1} k_{mj}\phi_m(x)\phi_j(\xi)$ *in* L^2 *with*

$$\sum_{j\geq 1} \lambda_j^{-1}|k_{mj}|^2 \to 0 \quad \text{and} \quad \sum_{m\geq 1} \lambda_m^{-1}|k_{mj}|^2 \to 0 \quad \text{fast enough}$$

Then: Observability, hence NC

Idea of the proof:

$$\begin{cases} -\varphi_t - \Delta\varphi + \int_\Omega K(\xi,x)\varphi(\xi,t)\,d\xi = 0 & \text{in } Q \\ \varphi = 0 & \text{on } \Sigma \\ \varphi(x,T) = \varphi^T(x) & \text{in } \Omega \end{cases}$$

$$\varphi = p + \zeta \text{, with } -p_t - \Delta p = 0, \text{ etc.}$$

We prove: $\|\varphi(\cdot\,,0)\|^2 \leq C \iint_{\omega\times(0,T)} |p|^2 \leq C \iint_{\omega\times(0,T)} |\varphi|^2$
We use (a) usual estimates of p and ζ and (b) compactness-uniqueness (and the properties of K!)

$$\begin{cases} y_t - a\left(\int_\Omega y(\xi,t)\,d\xi\right)\Delta y = v1_\omega \quad \text{in} \quad Q, \\ y = 0 \ \ \text{etc.} \end{cases}$$

Theorem (EFC-Limaco-Nina-Núñez, 2019)

Assume: $a \in C^2(\mathbf{R}),\ \ 0 < a_0 \leq a(r) \leq a_1$ *and* $|a'(r)| + |a''(r)| \leq M$,

$$\overline{y} : \ \textit{a free solution, } \overline{y}(\cdot\,,0) \in H^2 \cap H_0^1$$

Then: local EC to $\overline{y}$ *at time* T

Idea of the proof: Local NC of $z_t - \alpha_z(t)\Delta z + \beta_z(t)(\int_\Omega z)(-\Delta\overline{y}) = v1_\omega$ with $\alpha_z := a(\int_\Omega(z+\overline{y}))$, $\beta_z := a'(\int_\Omega(sz+\overline{y}))$

We use (a) a fixed-point argument and (b) the previous Theorem

$$\begin{cases} y_{tt} - \Delta y + \int_\Omega K(x,\xi) y(\xi,t)\, d\xi = v 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = z^0(x), \quad y_t(x,0) = z^1(x) & \text{in} \quad \Omega \end{cases}$$

NC? $\exists v$ such that $y(x,T) = 0$ and $y_t(x,T) = 0$ in Ω?

Theorem (EFC-Li-Zuazua, 2016)

Assume: K as before, classical GCC for (ω, T)
Then: NC

Similar (even easier) proof

Hence: EC (linear, reversible PDE)

Questions:

• Nonanalytic K?
Needed in the proofs to assert uniqueness

• Similar boundary control results?
Do usual extension, zero normal derivative arguments work?

• Systems with higher order nonlocal terms?
For instance: $y_t - \int_\Omega K(x,\xi)\Delta y(\xi,t)\,d\xi = v1_\omega$
Assumptions on K? [Andreu et al. 2010]

Also, systems with nonlocal nonlinear coefficients: [EFC et al 2012, Clark et al. 2013, Demarque-Limaco-Viana 2018, ...]

• Nonlocal nonlinear terms in the PDE?

Control of systems with nonlocal in time terms (memory)

Motivation: Memory effects on the physical behavior of a system, ...
The problem:

$$\left\{\begin{array}{l} u_t - \nu\Delta u - b\int_0^t e^{-a(t-s)}\Delta u(\cdot\,,s)\,ds + \nabla p = 0, \quad \nabla\cdot u = 0 \\ u = f 1_\gamma \ \text{ on } \ \partial\Omega\times(0,T) \\ u|_{t=0} = u_0, \quad \tau|_{t=0} = 0 \end{array}\right.$$

NC? $\exists f$ such that $u|_{t=T} = 0$?

- Viscoelastic (linearized) fluid in $\Omega\times(0,T)$, action on $\gamma\subset\partial\Omega$
  Blood, saliva, painting, polymers, ...
- Acting to drive to rest at $t = T$?
  OK if the fluid is only viscous ($b = 0$) or only elastic ($\nu = 0$)!

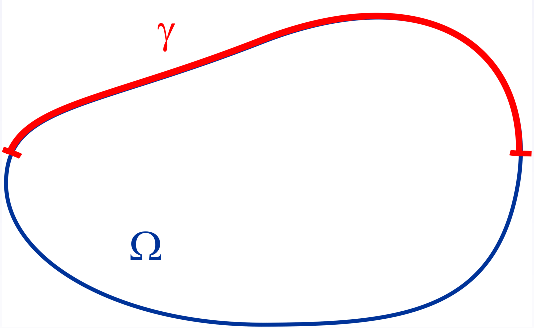


Figure: The domain and the active boundary

Figure: Viscoelasticity: painting

Figure: Viscoelasticity: saliva

Control of systems with memory

Reformulation:

$$\begin{cases} u_t - \nu\Delta u + \nabla p = \nabla\cdot\tau, \quad \nabla\cdot u = 0 \\ \tau_t + a\tau = bDu \\ u = f1_\gamma, \quad \text{etc.} \end{cases}$$

The linearized at zero of

$$\begin{cases} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \nabla\cdot\tau, \quad \nabla\cdot u = 0 \\ \tau_t + (u\cdot\nabla)\tau + a\tau + g(\nabla u,\tau) = b\,D(u) \\ u = f1_\gamma, \quad \text{etc.} \end{cases}$$

$$g(\nabla u,\tau) := \tau\cdot W(u) - W(u)\cdot\tau - \alpha(\tau\cdot D(u) + D(u)\cdot\tau), \quad \alpha\in[-1,1]$$

(viscoelastic fluid of the Oldroyd kind)

- Representing a non-Newtonian homogeneous fluid with memory ($\tau\equiv 0 \ \rightarrow$ Navier-Stokes)
- Much more complicate: $\exists$ global solution? (known for $\alpha = 0\dots$)
- $\exists$ stationary solutions only for large ν and a

[Guilopé-Saut, 1990; PL Lions-Masmoudi, 2000; EFC et al., 2002; . . .]

$$\begin{cases} u_t - \nu\Delta u - b\int_0^t e^{-a(t-s)}\Delta u(\cdot\,,s)\,ds + \nabla p = 0, \quad \nabla\cdot u = 0 \\ u = f 1_\gamma \quad \text{on} \quad \partial\Omega\times(0,T), \quad \text{etc.} \end{cases}$$

NC? $\forall u_0\ \exists f$ such that $u(\cdot\,,T) = 0$?

Theorem (Guerrero-Imanuvilov 2013, EFC et al. 2018, Renardy 2019)

No

Idea of the proof:

- NC $\Leftrightarrow$ Observability

$$\begin{cases} -\phi_t - \nu\Delta\phi + \int_t^T e^{-(s-t)}\Delta\phi(\cdot\,,s)\,ds + \nabla q = 0, \quad \nabla\cdot\phi = 0 \\ \phi = 0, \quad (x,t)\in\partial\Omega\times(0,T) \\ \phi(\cdot\,,T) = \phi^T \end{cases}$$

$$\|\phi(\cdot\,,0)\|_{L^2}^2 \le C\iint_{\omega\times(0,T)}|\phi|^2\,dx\,dt \quad \forall\phi^T$$

- But: $\exists\phi^{1,T},\phi^{2,T},\dots$ such that the associated ϕ^n satisfy
$$\iint_{\omega\times(0,T)}|\phi^n|^2\,dx\,dt \le C, \quad \|\phi^n(\cdot\,,0)\|_{L^2}^2 > n$$

Note: the answer is *Yes* if $b = 0$ (Stokes) or $\nu = 0$ (Maxwell) [Boldrini et al. 2012]

$$\left\{ \begin{array}{l} u_t - \nu\Delta u - b\int_0^t e^{-a(t-s)}\Delta u(\cdot\,,s)\,ds + \nabla p = 0, \quad \nabla\cdot u = 0 \\ u = f1_\gamma \quad \text{on} \quad \partial\Omega\times(0,T), \quad \text{etc.} \end{array} \right.$$

AC? $\forall u_0, u_1,\ \forall \varepsilon > 0\ \exists f_\varepsilon$ such that $\|u_\varepsilon(\cdot\,,T) - u_1\|_{L^2} \le \varepsilon$?

Theorem (Doubova-EFC 2012)

Yes

Idea of the proof:

- AC $\Leftrightarrow$ Unique continuation

$$\left\{ \begin{array}{l} -\phi_t - \nu\Delta\phi + \int_t^T e^{-(s-t)}\Delta\phi(\cdot\,,s)\,ds = 0, \quad \nabla\cdot\phi = 0 \\ \phi = 0, \quad (x,t)\in\omega\times(0,T) \end{array} \right. \quad \Rightarrow \quad \phi \equiv 0$$

- From the structure and properties of ϕ:
  1. $\exists$ analytical extension to $\{z\in\mathbb{C}\ :\ \mathrm{Re}\, z < T\}$
  2. $\zeta := (\phi(\cdot\,,t),\psi)_{L^2} \equiv 0$ (and $\tilde\zeta \equiv 0$) if $\psi = 0$ outside ω

  The consequence: $\phi \equiv 0$

A previous similar result: [Brandao et al., 2009]

Other results, final remarks and open questions:

- Moving controls + GCC lead to NC. Proved for some “simplified” systems [Chaves et al. 2014-...]

$$u_t - \nu\Delta u + mu = z, \quad z_t + az = bu + f1_{\omega(t)} \quad \text{etc.}$$

Extensions for linearized Oldroyd?

- Also for wave PDEs with memory [Biccari-Micu, 2019]
- Boundary moving controls?
- NC for the original nonlinear system? AC? NC with moving controls?

$$\begin{cases} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \nabla\cdot\tau + f1_{\omega(t)}, \quad \nabla\cdot u = 0 \\ \tau_t + (u\cdot\nabla)\tau + a\tau + g(\nabla u,\tau) = b\,D(u) \\ \text{etc.} \end{cases}$$

Recall [Coron-Lissy 2014] ...

- Numerical results? For moving controls?
For approximate controls? Note: $\|f_\varepsilon 1_\omega\|_{L^\infty} \to \infty$

A last control problem concerning epidemics

Age-structured SIR model (generalized from [Kermack-McKendrick 1927])

$$S = S(t), i = i(a,t), R = R(t)$$

a is the class-age: time elapsed since infection

$$\begin{cases} S_t = -\frac{1}{N}\left(\int_0^{a_*}\left[\beta(a) - (f1_\omega)(a,t)\right] i(a,t)\,da\right) S & \text{in} \quad (0,T) \\ i_t + i_a = -\gamma(a) i & \text{in} \quad (0,a_*)\times(0,T) \\ i\big|_{a=0} = \frac{1}{N}\left(\int_0^{a_*}\left[\beta(a) - (f1_\omega)(a,t)\right] i(a,t)\,da\right) S & \text{in} \quad (0,T) \\ S\big|_{t=0} = S_0, \quad i\big|_{t=0} = i_0(a) & \text{in} \quad (0,a_*) \\ R(t) = N - S(t) - \int_0^{a_*} i(a,t)\,da & \text{in} \quad (0,T) \end{cases}$$

An “academic” control problem:

Find “admissible” $f = f(a,t)$ (and ω) such that $S(T) = S_*$, $i(a,T) = i_*$ in $(0, a_*)$

$f1_\omega$ represents vaccination and isolation actions

Boundary, bilinear, nonlinear state system . . .

Other models and other control results: [Iannelli-Milner 2017, Kokomo et al. 2020, Grigorieva et al. 2020, . . .]

THANK YOU VERY MUCH . . .

MUCHAS GRACIAS POR VUESTRA ATENCIÓN . . .