Hierarchical Control Problems for PDEs

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Control en Tiempos de Crisis

João Pessoa - Brasil
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Standard Controllability Problems

\[
\begin{align*}
&y_t - Ay = f \quad \text{for} \quad t \in [0, T], \\
&y|_{t=0} = y_0.
\end{align*}
\]
Standard Controllability Problems

\[ \begin{align*} \{ & y_t - Ay = f \quad \text{for} \quad t \in [0, T], \\ & y|_{t=0} = y_0. \end{align*} \]

Approximate Controllability

For every $y_0$ and $y_1$, there exists a control $f$ such that $y(T)$ is “close” to $y_1$. 
Standard Controllability Problems

\[
\begin{aligned}
\left\{\begin{array}{l}
y_t - Ay = f \quad \text{for} \quad t \in [0, T], \\
y|_{t=0} = y_0.
\end{array}\right.
\end{aligned}
\]

Exact Controllability

For every \( y_0 \) and \( y_1 \) there exists a control \( f \) such that \( y(T) = y_1 \).
Standard Controllability Problems

\[
\begin{cases}
  y_t - Ay = f & \text{for } t \in [0, T], \\
  y|_{t=0} = y_0.
\end{cases}
\]

Null Controllability

For every $y_0$, there exists a control $f$ such that $y(T) = 0$. 

![Graph showing null controllability](image-url)
Multi-objective control problem

Mono-objective:
\[ Ly = f^1 \big|_\Omega, \quad \Omega \times (0, T) \]
\[ y(0) = y^0, \quad y(T) = y^T, \quad \Omega \]
Multi-objective control problem

\[ \Omega \]

\[ \begin{align*}
\text{Mono-objective:} & \\
Ly &= f^1|_\Omega, & \Omega \times (0,T) \\
y(0) &= y^0, & y(T) = y^T, & \Omega \\
\end{align*} \]

\[ \begin{align*}
\text{Multi-objective:} & \\
Ly &= f^1|_\Omega + v^1_1|_\Omega_1 + v^1_2|_\Omega_2, & \Omega \times (0,T) \\
y(0) &= y^0, & y(T) = y^T, & \Omega \\
y & \simeq y_{i,d}, & \Omega_1 & \Omega_2 \\
\end{align*} \]
Multi-objective control problem

Mono-objective:
\[ Ly = f_{1|\Omega}, \quad \Omega \times (0, T) \]
\[ y(0) = y^0, \quad y(T) = y^T, \quad \Omega \]
\[ y \simeq y_{i,d}, \quad O_i,d \]

Multi-objective:
\[ Ly = f_{1|\Omega} + v_{1|\Omega_1} + v_{2|\Omega_2}, \quad \Omega \times (0, T) \]
\[ y(0) = y^0, \quad y(T) = y^T, \quad \Omega \]
\[ y \simeq y_{i,d}, \quad O_i,d \]
Temperature Distribution
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Let $y$ be the concentration of a chemical product in a lake;

We want to drive this concentration to a prescribed value in a final time $T$;

In some regions of the lake, you cannot have the concentration far from some given values.
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- We want to drive this concentration to a prescribed value in a final time \( T \);
- In some regions of the lake, you cannot have the concentration far from some given values.
The Black Scholes Equation

\[
\frac{\partial y}{\partial t} - x^2 \frac{\partial^2 y}{\partial x^2} + s \frac{\partial y}{\partial x} - ky = 0
\]

- The price evolution of European call options;
- A contract that gives the buyer the right, but not the obligation, to buy an specific asset;
- Many agents may be interested at the same time to change the price of the option.
SIR model with vital dynamics and spatial heterogeneity

\[
\begin{aligned}
\frac{\partial s}{\partial t} &= d_1 \Delta s - k i s + \mu (1 - s) \quad \text{in} \quad \Omega \times (0, T), \\
\frac{\partial i}{\partial t} &= d_2 \Delta i + k i s - (\mu + \gamma) i \quad \text{in} \quad \Omega \times (0, T), \\
\frac{\partial r}{\partial t} &= d_3 \Delta r + k r s - \mu r \quad \text{in} \quad \Omega \times (0, T).
\end{aligned}
\]

- \(s\) represents the density of susceptible individuals.
- \(i\) represents the density of infective individuals.
- \(r\) represents the density of removed individuals.
**Figure:** Distribution of susceptible and infective individuals
SIR Model

[Map of João Pessoa with circles labeled $O_1, d$, $O_2, d$, and $O_3, d$.]
The strategy

- We assume a hierarchy between the controls, one leader and two (or more) followers.
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- The leader establish his strategy and the followers must adapt themselves to accomplish their objectives in an optimal way.
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- The leader establish his strategy and the followers must adapt themselves to accomplish their objectives in an optimal way.
- The leader wants to drive the solution exactly to a prescribed state in a final time

\[ y(T) = y^T. \]
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- The leader establish his strategy and the followers must adapt themselves to accomplish their objectives in an optimal way.
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  \[ y(T) = y^T. \]
- The followers wants to make the solution “not far” from some prescribed functions \( y_{i,d} \).
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- The leader establish his strategy and the followers must adapt themselves to accomplish their objectives in an optimal way.
- The leader wants to drive the solution exactly to a prescribed state in a final time
  \[ y(T) = y^T. \]
- The followers wants to make the solution “not far” from some prescribed functions \( y_{i,d} \).
- Mainly, we use the concept of Nash equilibrium to determine the followers.
Classical references:


Description of the method:

Consider the equation:

\[
\begin{aligned}
& y_t - \Delta y + a(x, t)y = f1_\Omega + v^11_{\mathcal{O}_1} + v^21_{\mathcal{O}_2} \quad \text{in} \quad Q, \\
& y = 0 \quad \text{on} \quad \Sigma, \\
& y(x, 0) = y^0(x) \in L^2(\Omega) \quad \text{in} \quad \Omega,
\end{aligned}
\]

and the functionals

\[
J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_i, x \times (0, T)} |y - y_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0, T)} |v^i|^2 \, dx \, dt.
\]
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\begin{cases}
  y_t - \Delta y + a(x, t)y = f1_\mathcal{O} + v^11_{\mathcal{O}_1} + v^21_{\mathcal{O}_2} & \text{in } Q, \\
  y = 0 & \text{on } \Sigma, \\
  y(x, 0) = y^0(x) \in L^2(\Omega) & \text{in } \Omega,
\end{cases}
\]

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J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_i \times (0, T)} |y - y_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0, T)} |v^i|^2 \, dx \, dt.
\]

Objective

Find \( f \) such that \( y(T) = 0 \). For each \( f \), the pair \( (v^1(f), v^2(f)) \) is a Nash equilibrium associated to the functionals \( J_i \).

- \( f \in L^2(\mathcal{O} \times (0, T)) \) is the leader control
- \( v^i \in L^2(\mathcal{O}_i \times (0, T)) \) are the followers
- \( \alpha_i > 0, \mu_i >> 1 \) and \( y_{i,d} \) are regular functions
Nash Equilibrium

\[
\begin{align*}
J_1(f; v^1, v^2) &= \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2) \\
J_2(f; v^1, v^2) &= \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2)
\end{align*}
\]
Nash Equilibrium

\[
\begin{cases}
J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2) \\
J_2(f; v^1, v^2) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2)
\end{cases}
\]

Nash Quasi Equilibrium

\[
\begin{cases}
J_1'(f; v^1, v^2)(\hat{v}^1, 0) = 0 \quad \forall \hat{v}^1 \in L^2(\mathcal{O}_1 \times (0, T)), \\
J_2'(f; v^1, v^2)(0, \hat{v}^2) = 0, \quad \forall \hat{v}^2 \in L^2(\mathcal{O}_2 \times (0, T)).
\end{cases}
\]

**Remark:** Since the problem is linear and the functionals are convex in each direction, the equilibrium and quasi equilibrium concepts are equivalents.
The Nash equilibrium \((v^1, v^2)\) is characterized by the following optimality system:

\[
\begin{aligned}
&y_t - \Delta y + a(x, t)y = f_1 - \frac{1}{\mu_1} \phi^1_1 - \frac{1}{\mu_2} \phi^2_1 \quad \text{in} \quad Q, \\
&-\phi^i_t - \Delta \phi^i + a(x, t)\phi^i = \alpha_i(y - y_{i,d})_1 \quad \text{in} \quad Q, \\
&y = 0, \quad \phi^i = 0 \quad \text{on} \quad \Sigma, \\
&y(x, 0) = y^0(x), \quad \phi^i(x, T) = 0 \quad \text{in} \quad \Omega.
\end{aligned}
\]

Precisely, we have that

\[
\begin{align*}
v_1 &= -\frac{1}{\mu_1} \phi^1 \\
v_2 &= -\frac{1}{\mu_2} \phi^2
\end{align*}
\]

Can we find \(f\) such that \(y(T) = 0\)?
The controllability result is equivalent to the following observability inequality (HUM)

\[
\int_{\Omega} |\psi(x,0)|^2 \, dx + \sum_{i=1}^{2} \int_{Q} \hat{\rho}^{-2} |\gamma^i|^2 \, dx \, dt \leq C \int_{\Omega \times (0,T)} |\psi|^2 \, dx \, dt.
\]

Adjoint System

\[
\begin{aligned}
-\psi_t - \Delta \psi + a(x,t)\psi &= \alpha_1 \gamma^1 \mathbf{1}_{O_1} d + \alpha_2 \gamma^2 \mathbf{1}_{O_2} d & \text{in} & \quad Q, \\
\gamma^i_t - \Delta \gamma^i &= -\frac{1}{\mu_i} \psi \mathbf{1}_{O_i} & \text{in} & \quad Q, \\
\psi &= 0, \quad \gamma^i = 0 & \text{on} & \quad \Sigma, \\
\psi(x,T) &= \psi^T, \quad \gamma^i(x,0) = 0 & \text{in} & \quad \Omega.
\end{aligned}
\]
Assume that

- $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$
- $\int\int_{\mathcal{O}_d \times (0,T)} \hat{\rho}^2 |y_{i,d}|^2 \, dx \, dt < +\infty$ where $\lim_{t \to T} \hat{\rho} = +\infty$

For each $y_0 \in L^2(\Omega)$, there exist controls $f \in L^2(\mathcal{O} \times (0,T))$ and a Nash equilibrium $(v^1, v^2)$ such that $y(x, T) = 0$ in $\Omega$. 
Sketch of the proof

- We prove a suitable Carleman for the adjoint system.

\[ I(\psi, h) \leq C \int_\omega \int_0^T (\rho_1^{-2}|\psi|^2 + \rho_2^{-2}|h|^2) \, dx \, dt, \]

- \( h = \alpha_1 \gamma_1 + \alpha_2 \gamma_2; \)
- \( I(\psi, h) \) is a weighted \( H^1 \) norms;
- \( \rho_1^{-2} \) and \( \rho_2^{-2} \) are weight functions blowing up at \( t = T. \)
Sketch of the proof

- We prove a suitable Carleman for the adjoint system.

\[ I(\psi, h) \leq C \int \int_{\omega \times (0,T)} (\rho_1^{-2}|\psi|^2 + \rho_2^{-2}|h|^2) \, dx \, dt, \]

- \( h = \alpha_1 \gamma_1 + \alpha_2 \gamma_2; \)
- \( I(\psi, h) \) is a weighted \( H^1 \) norms;
- \( \rho_1^{-2} \) and \( \rho_2^{-2} \) are weight functions blowing up at \( t = T. \)
- We prove that

\[ \int \int_{\omega \times (0,T)} \rho_2^{-2}|h|^2 \, dx \, dt \, dx \, dt \leq \int \int_{\omega_1 \times (0,T)} \rho_3^{-2}|\psi|^2 \, dx \, dt \, dx \, dt \]

Here, we use that
- \( \omega \subseteq \omega_1. \)
- \(-\psi_t - \Delta \psi + a(x, t)\psi = \alpha_1 \gamma^1 1_{\omega_1} + \alpha_2 \gamma^2 1_{\omega_2} \) in \( \omega. \)
Assume that

1. $O_{1,d} \cap O \neq O_{2,d} \cap O$
2. $\int \int_{O_d \times (0,T)} \hat{\rho}^2 |y_{i,d}|^2 \, dx \, dt < +\infty$ where $\lim_{t \to T} \hat{\rho} = +\infty$

For each $y_0 \in L^2(\Omega)$, there exist controls $f \in L^2(\mathcal{O} \times (0, T))$ and a Nash quasi equilibrium $(v^1, v^2)$ such that the solution (15) satisfies

$$y(x, T) = 0 \quad \text{in} \quad \Omega. \quad (1)$$
We prove a suitable Carleman for the adjoint system.

\[
I(\psi, \gamma^1, \gamma^2) \leq C \int\int_{\omega \times (0,T)} \rho_1^{-2} |\psi|^2 \, dx \, dt + \text{RHS}
\]

\[
+ \sum_{i=1}^{2} \int\int_{\omega_i \times (0,T)} \rho_i^{-2} |\gamma^i|^2 \, dx \, dt \, dx \, dt , \quad (2)
\]
We prove a suitable Carleman for the adjoint system.

\[ I(\psi, \gamma^1, \gamma^2) \leq C \int\int_{\omega \times (0,T)} \rho_1^{-2} |\psi|^2 \, dx \, dt + \text{RHS} \]
\[ + \sum_{i=1}^{2} \int\int_{\omega_i \times (0,T)} \rho_i^{-2} |\gamma^i|^2 \, dx \, dt \, dx \, dt, \quad (2) \]

- \( I(\psi, \gamma_1, \gamma_2) \) is a weighted \( L^2 \) norm for \( \psi \) and \( H^2 \) for \( \gamma_i \);
- If we use the equation directly we obtain
  \[ \alpha_1 \gamma^1_1 \sigma_1 \, d = -\psi_t - \Delta \psi + a(x,t)\psi - \alpha_2 \gamma^2_1 \sigma_2 \, d \]
  or
  \[ \alpha_2 \gamma^2_2 \sigma_2 \, d = -\psi_t - \Delta \psi + a(x,t)\psi - \alpha_1 \gamma^1_1 \sigma_1 \, d \]
- If we try to remove \( \gamma^1 \), \( \psi \) (Good!) and \( \gamma^2 \) (Bad!) appears.
- If we try to remove \( \gamma^2 \), \( \psi \) (Good!) and \( \gamma^1 \) (Bad!) appears.
To solve this we choose the sets $\omega_i$, $i = 1, 2$ such that

$$\omega_1 \subset O_{1,d} \cap O_{2,d}^c$$

and

$$\omega_2 \subset O_{2,d} \cap O_{1,d}^c$$

After that we construct suitable weight functions to absorb some global terms.
Figure: Weight Functions
Geometry:

- 2015

\[ O_{1,d} = O_{2,d} = O_d \]

- 2017
Consider the KS equation

\[
\begin{align*}
    & y_t + y_{xxxx} + \nu y_{xx} + yy_x = f_1 \varnothing + v^1_1 \varnothing_1 + v^2_1 \varnothing_2 & \text{in } Q, \\
    & y(0, t) = y(L, t) = 0 & \text{in } (0, T), \\
    & y_x(0, t) = y_x(L, t) = 0 & \text{in } (0, T), \\
    & y(x, 0) = y^0(x) & \text{in } (0, L).
\end{align*}
\]

- Phase turbulence in reaction-diffusion systems
- Plane flame propagation

**Objective**

Find \( f \) such that \( y(T) = 0 \). For each \( f \), the pair \((v^1(f), v^2(f))\) is a *Nash equilibrium* associated to the functionals \( J_i \).

\[
J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \int_{\varnothing_i \times (0, T)} |y - y_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \int_{\varnothing_i \times (0, T)} |v^i|^2 \, dx \, dt.
\]
Suppose
\[ O_{i,d} \cap \Omega \neq \emptyset \quad (i = 1, 2), \]
and that one of the following two conditions holds:
\[ O_{1,d} = O_{2,d} \quad \text{or} \quad O_{1,d} \cap \Omega \neq O_{2,d} \cap \Omega. \]

Let \( \bar{y} \) is a sufficiently regular trajectory. There exists \( \delta > 0 \) and positive functions \( \hat{\rho}_i = \hat{\rho}_i(t) \) blowing up at \( t = T \) such that if \( y^0 \) and \( \bar{y} \) satisfy
\[
\|y^0 - \bar{y}^0\|_{L^2(0,L)}^2 + \sum_{i=1,2} \int_0^T \int_{O_{i,d}} \hat{\rho}_i^2 |\bar{y} - y_{i,d}|^2 \, dx \, dt < \delta,
\]
there exist controls \( f \in L^2(\Omega \times (0,T)) \) and associated Nash equilibria \( (v^1, v^2) \) such that \( y(T) = \bar{y}(T) \).
The Nash equilibrium \((v^1, v^2)\) is characterized by the following optimality system:

\[
\begin{aligned}
&z_t + z_{xxx} + \nu z_{xx} + zz + (\bar{y}z)_x = f \quad \text{in} \ Q, \\
&-\phi_t^i + \phi_{xxx}^i + \nu \phi_{xx}^i - (z + \bar{y})\phi_x^i = \alpha_i(z - z_{i,d}) \quad \text{in} \ Q, \\
&z(0,t) = z(L,t) = \phi_x^i(0,t) = \phi_x^i(L,t) = 0 \quad \text{in} \ (0,T), \\
&z_x(0,t) = z_x(L,t) = \phi_{x}^i(0,t) = \phi_{x}^i(L,t) = 0 \quad \text{in} \ (0,T), \\
&z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 \quad \text{in} \ (0,L).
\end{aligned}
\]

Precisely, we have that

\[
v_1 = -\frac{1}{\mu_1} \phi^1 \quad \text{and} \quad v_2 = -\frac{1}{\mu_2} \phi^2
\]

Can we find \(f\) such that \(y(T) = 0\)?

- Nonlinear Problem;
- Non convex functionals;
- Nash and quasi Nash equilibrium are equivalent for \(\mu_1, \mu_2\) large.
Observability inequality

\[ \int_{\Omega} |\psi(x, 0)|^2 \, dx + \sum_{i=1}^{2} \int \int_{Q} \hat{\rho}^{-2} |\gamma^i|^2 \, dx \, dt \leq C \int \int_{\mathcal{O} \times (0, T)} |\psi|^2 \, dx \, dt. \]

Adjoint System

\[
\begin{aligned}
-\psi_t + \psi_{xxxx} + \nu \psi_{xx} - \bar{y} \psi_x &= g^0 + \alpha_1 \gamma^1_{1\mathcal{O}_{1,d}} + \alpha_2 \gamma^2_{1\mathcal{O}_{2,d}} \quad \text{in } Q, \\
\gamma^i_t + \gamma^i_{xxxx} + \nu \gamma^i_{xx} + (\bar{y} \gamma^i)_x &= g^i - \frac{1}{\mu_i} \psi_{1\mathcal{O}_i} \quad i = 1, 2 \quad \text{in } Q, \\
\psi(0, t) = \psi(L, t) = \gamma^i(0, t) = \gamma^i(L, t) &= 0 \quad i = 1, 2 \quad \text{in } (0, T), \\
\psi_x(0, t) = \psi_x(L, t) = \gamma^i_x(0, t) = \gamma^i_x(L, t) &= 0 \quad i = 1, 2 \quad \text{in } (0, T), \\
\psi(x, T) = \psi^T(x), \quad \gamma^i(x, 0) &= 0 \quad i = 1, 2 \quad \text{in } (0, L). 
\end{aligned}
\]
Main Ideas

- We start proving

\[ I(\psi, \gamma^1, \gamma^2) \leq C \iiint_{\omega \times (0,T)} \rho_1^{-2} |\psi|^2 \, dxdt + RHS \]

\[ + \sum_{i=1}^{2} \iiint_{\omega_i \times (0,T)} \rho_i^{-2} |\gamma^i|^2 \, dxdt \, dxdt, \quad (3) \]
Main Ideas

- We start proving

\[ I(\psi, \gamma^1, \gamma^2) \leq C \iiint_{\omega \times (0,T)} \rho_1^{-2} |\psi|^2 \, dxdt + \text{RHS} \]

\[ + \sum_{i=1}^{2} \iiint_{\omega_i \times (0,T)} \rho_i^{-2} |\gamma^i|^2 \, dxdtdxdt, \quad (3) \]

- \( I(\psi, \gamma_1, \gamma_2) \) is a weighted \( L^2 \) norm for \( \psi \) and \( H^4 \) for \( \gamma_i \);
- To absorb RHS, we prove a Carleman estimate with right-hand side in \( H^{-4} \);
- The local terms of \( \gamma_i \) are absorbed using the equation

\[ \gamma^i = \frac{1}{\alpha_i} (-\psi_t + \psi_{xxxx} + \nu \psi_{xx} - \bar{y}\psi_x - g^0) \quad \text{in} \ \omega_i \times (0,T), \quad i = 1, 2. \]
The Korteweg-de Vries Equation

Consider the KdV equation

\[
\begin{cases}
y_t + y_x + y_{xxx} + y y_x = f^1 \mathcal{O} + v^1 \chi \mathcal{O}_1 + v^2 \chi \mathcal{O}_2 & \text{in } Q, \\
y(0, \cdot) = y(L, \cdot) = y_x (L, \cdot) = 0 & \text{on } (0, T), \\
y(x, \cdot) = y^0 & \text{in } (0, 1),
\end{cases}
\]

- Models the propagation of water waves on shallow water surfaces

**Objective**

Find \( f \) such that \( y(T) = 0 \). For each \( f \), the pair \( (v^1(f), v^2(f)) \) is a *Nash equilibrium* associated to the functionals \( J_i \).

\[
J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \int \int_{\mathcal{O}_i \times (0, T)} |y - y_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \int \int_{\mathcal{O}_i \times (0, T)} |v^i|^2 \, dx \, dt.
\]
Suppose
\[ O_{i,d} \cap O \neq \emptyset \quad (i = 1, 2), \]
and that one of the following two conditions holds:
\[ O_{1,d} = O_{2,d} \quad \text{or} \quad O_{1,d} \cap O \neq O_{2,d} \cap O. \]

Let \( \bar{y} \) is a sufficiently regular trajectory. There exists \( \delta > 0 \) and positive functions \( \hat{\rho}_i = \hat{\rho}_i(t) \) blowing up at \( t = T \) such that if \( y^0 \) and \( \bar{y} \) satisfy
\[
\|y^0 - \bar{y}^0\|_{L^2(0,L)}^2 + \sum_{i=1,2} \int_0^T \int_{O_{i,d}} \hat{\rho}_i^2 |\bar{y} - y_{i,d}|^2 \, dx \, dt < \delta,
\]
there exist controls \( f \in L^2(O \times (0, T)) \) and associated Nash equilibria \( (v^1, v^2) \) such that \( y(T) = \bar{y}(T) \).
The Nash equilibrium \((v^1, v^2)\) is characterized by the following optimality system:

\[
\begin{cases}
  z_t + z_x + z_{xxx} + zz_x + (yz)_x = f 1_\Omega - \sum_{i=1}^2 \frac{1}{\mu_i} \phi^i \chi_{\Omega_i} \\
  -\phi_t^i - \phi_x^i - \phi_{xxx}^i - (z + \bar{y})\phi_x^i = \alpha_i (z - z_i,d) \chi_{\Omega_i,d}, \ i = 1, 2 \\
  z(0, \cdot) = z(L, \cdot) = z_x(L, \cdot) = 0 \\
  \phi^i(0, \cdot) = \phi^i(L, \cdot) = \phi_x^i(0, \cdot) = 0 \\
  z(\cdot, 0) = z^0, \ \phi^i(\cdot, T) = 0
\end{cases}
\]

in \(Q\), \(Q\), \((0, T)\), \((0, T)\), \((0, 1)\).

Precisely, we have that

\[ v_1 = -\frac{1}{\mu_1} \phi^1 \text{ and } v_2 = -\frac{1}{\mu_2} \phi^2 \]

Can we find \(f\) such that \(z(T) = 0\)?

- Nonlinear Problem;
- Non convex functionals;
- Nash and quasi Nash equilibrium are equivalent for \(\mu_1, \mu_2\) large.
**Observability Inequality**

\[ \int_{\Omega} |\psi(x, 0)|^2 \, dx + \sum_{i=1}^{2} \iint_{Q} \hat{\rho}^{-2} |\gamma^i|^2 \, dx \, dt \leq C \iint_{\Omega \times (0, T)} |\psi|^2 \, dx \, dt. \]

**Adjoint System**

\[
\begin{cases}
-\psi_t - M\psi_x - \psi_{xxx} = \sum_{i=1}^{2} \alpha_i \gamma^i \chi_{O_i,d} + g^0 & \text{in } Q, \\
\gamma^i_t + (M\gamma^i)_x + \gamma^i_{xxx} = -\frac{1}{\mu_i} \psi \chi_{O_i} + g^i, & 1 = 1, 2 \text{ in } Q, \\
\psi(0, \cdot) = \psi(L, \cdot) = \psi_x(0, \cdot) = 0 & \text{on } (0, T), \\
\gamma^i(0, \cdot) = \gamma^i(L, \cdot) = \gamma^i_x(L, \cdot) = 0 & \text{on } (0, T), \\
\psi(x, T) = \psi^T(x), \quad \gamma^i(\cdot, 0) = 0 & \text{in } (0, 1). 
\end{cases}
\]
Main Ideas

- We start proving

\[
I(\psi, \gamma^1, \gamma^2) \leq C \int_0^T \left( \rho_1^{-2} \left( |\psi|^2 + |\psi_{xx}|^2 \right) + \rho_i^{-2} \left( |\gamma_i|^2 + |\gamma_{xx}^i|^2 \right) \right) \, dx \, dt + RHS
\]

\[
+ \sum_{i=1}^{2} \int_0^T \left( \rho_1^{-2} \left( |\psi|^2 + |\psi_{xx}|^2 \right) + \rho_i^{-2} \left( |\gamma_i|^2 + |\gamma_{xx}^i|^2 \right) \right) \, dx \, dt,
\]

\( I(\psi, \gamma^1, \gamma^2) \) is a weighted \( H_2 \) norm for \( \psi \) and for \( \gamma_i \); (Not good!)

To absorb RHS and the local terms of second order derivative, we use energy estimates to add a high-order term on the left-hand side; (\( H_8/3 \))

The local terms of \( \gamma_i \) are absorbed using the equation

\[-\psi_t - M \psi_x - \psi_{xxx} - g_0 = \alpha_i \gamma_i \chi_{O_i,d} \in \omega_i \times (0,T), \]

for \( i = 1, 2 \).
Main Ideas

- We start proving

\[
I(\psi, \gamma^1, \gamma^2) \leq C \iint_{\omega \times (0,T)} \rho_1^{-2} \left( |\psi|^2 + |\psi_{xx}|^2 \right) \, dx \, dt + RHS
\]
\[
+ \sum_{i=1}^{2} \iint_{\omega_i \times (0,T)} \rho_i^{-2} \left( |\gamma^i|^2 + |\gamma^i_{xx}|^2 \right) \, dx \, dt,
\]

- \( I(\psi, \gamma^1, \gamma^2) \) is a weighted \( H^2 \) norm for \( \psi \) and for \( \gamma^i \); (Not good!)

- To absorb RHS and the local terms of second order derivative, we use energy estimates to add a high-order term on the left-hand side; \( H^{8/3} \)

- The local terms of \( \gamma^i \) are absorbed using the equation

\[
-\psi_t - M\psi_x - \psi_{xxx} - g^0 = \alpha_i \gamma^i \chi_{O_i,d} \quad \text{in} \quad \omega_i \times (0,T), \quad \text{for} \quad i = 1, 2.
\]
Open Questions

The Navier-Stokes equation

\[
\begin{cases}
  y_t - \Delta y + (y \cdot \nabla)y + \nabla p = f1_Ω + v^1 1_{Ω_1} + v^2 1_{Ω_2} & \text{in } Q, \\
  \nabla \cdot y = 0 & \text{in } Q, \\
  y = 0 & \text{on } \Sigma, \\
  y(x, 0) = y^0(x) & \text{in } Ω.
\end{cases}
\]

- Much more complex than the Heat equation case;
- The weight functions are difficult to build.
• Stackelberg-Pareto Strategy
  There is no \((\hat{u}^1, \hat{u}^2) \in \mathcal{H}\) satisfying

  \[
  J_i(\hat{u}^1, \hat{u}^2) \leq J_i(u^1(f), u^2(f)), \quad i = 1, 2,
  \]

  one of these inequalities at least being strict.

  • Optimal pairs solves the minimization problem

  \[
  \min_{v_1, v_2} \left( (1 - \lambda)J_1(v_1, v_2) + \lambda J_2(v_1, v_2) \right),
  \]

  for each \(\lambda \in (0, 1)\).
- What about this case?
Thank You for Your Attention!